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# Can the tidal quality factors of terrestrial planets and moons scale as positive powers of the tidal frequency?

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**Abstract**

In geophysics and seismology, it is a common knowledge that the quality factors $Q$ of the mantle and crust materials scale as the tidal frequency to a positive fractional power (Karato 2007, Efroimsky & Lainey 2007). In astronomy, there exists an equally common belief that such rheological models introduce discontinuities into the equations and thus are unrealistic at low frequencies. We demonstrate that, while such models indeed make the conventional expressions for the tidal torque diverge for vanishing frequencies, the emerging infinities reveal not the impossible nature of one or another rheology, but a subtle flaw in the underlying mathematical model of friction. Flawed is the common misassumption that the tidal force and torque are inversely proportional to the quality factor. In reality, they are proportional to the sinus of the tidal phase lag. This quantity is very close to the inverse quality factor everywhere *except in the vicinity of the zero frequency*. Reinstating of this detail tames the fake infinities and rehabilitates the "impossible" scaling law (which happens to be the actual law the mantles obey).

This preprint is a pilot paper. A more comprehensive treatise on tidal torques is to be published (Efroimsky & Williams 2009).

# 1 Introduction.

We are considering bodily tides in a primary perturbed by a secondary. Each elementary volume of the primary is subject to a tide-raising potential, which in general is not periodic but can be expanded into a sum of periodic terms. We shall assume that the primary is homogeneous and incompressible. Although simplistic, the model provides a good qualitative understanding of tidal evolution of both the primary's spin and the secondary's orbit

# 2 Linearity of the tide

## 2.1 Two aspects of linearity

We assert deformations to be linear. Each tidal harmonic $W_l$ of the potential disturbance produced by the secondary generates a linear deformation of the primary's shape, while each such deformation amends the potential of the primary with an addition proportional to the Love number $k_l$. Linearity also implies that the energy attenuation rate $\langle\dot{E}\;(\chi)\rangle$ at each frequency $\chi$ depends solely on the value of $\chi$ and on the amplitude $E_{peak}(\chi)$, and is not influenced by the other harmonics. Thus,

$$\langle\,\dot{E}(\chi)\,\rangle\;=\;-\;\frac{\chi E_{peak}(\chi)}{Q(\chi)} \tag{1}$$

or, equivalently:

$$\Delta E_{cycle}(\chi)\;=\;-\;\frac{2\;\pi\;E_{peak}(\chi)}{Q(\chi)}\quad, \tag{2}$$

$\Delta E_{cycle}(\chi)$ being the one-cycle energy loss. The so-defined quality factor $Q(\chi)$ corresponding to some frequency $\chi$ is interconnected with the phase lag $\epsilon(\chi)$ corresponding to the same frequency.

If $E_{peak}(\chi)$ in (1 - 2) is agreed to denote the peak *energy* stored at frequency $\chi$, the appropriate $Q$ factor is connected to the phase lag $\epsilon(\chi)$ through

$$Q^{-1}_{energy}\;=\;\tan|\epsilon|\quad. \tag{3}$$

If $E_{peak}(\chi)$ is defined as the peak *work*, the corresponding $Q$ factor is related to the lag via

$$Q^{-1}_{work}\;=\;\frac{\tan|\epsilon|}{1\;-\;\left(\;\frac{\pi}{2}\;-\;|\epsilon|\;\right)\;\tan|\epsilon|}\quad, \tag{4}$$

as demonstrated in the Appendix below.

Both definitions render a vanishing $Q$ for the lag approaching $\pi/2$, and both result in the same approximation for $Q$ in the limit of a small lag:

$$Q^{-1}_{energy}\;=\;\sin|\epsilon|\;+\;O(\epsilon^2)\;=\;|\epsilon|\;+\;O(\epsilon^2)\quad. \tag{5}$$

Hence definitions (3 - 4) make $1/Q$ a good approximation to $\sin\epsilon$ for small lags only.

This makes the essence of the standard, linear theory of bodily tides. The model permits for the freedom of choice of the functional dependency of the quality factor upon the tidal frequency. Whatever the form of this dependence, the basic idea of the theory is the following: the tide-raising potential is expanded over periodic terms, whereafter (a) the material's response is assumed to be linear, and (b) the overall attenuation rate is asserted to be a sum of rates corresponding to the involved frequencies.

## 2.2 Goldreich's admonition: a general difficulty stemming from nonlinearity

Introduced empirically as a means to figleaf our lack of knowledge of the attenuation process in its full complexity, the notion of $Q$ has proven to be practical due to its smooth and universal dependence upon the frequency and temperature. At the same time, this empirical treatment has its predicaments and limitations. Its major inborn defect was brought to light by Peter Goldreich who pointed out that the attenuation rate at a particular frequency depends not only upon the appropriate Fourier component of the stress, but also upon the *overall* stress. This happens because for real minerals each quality factor $Q(\chi_i)$ bears dependence not only on the frequency $\chi_i$ itself, but also on the magnitude of the $\chi_i-$component of the stress and, most importantly, also on the *overall* stress. This, often-neglected, manifestation of nonlinearity may be tolerated only when the amplitudes of different harmonics of stress are comparable. However, when the amplitude of the principal mode is orders of magnitude higher than that of the harmonics (tides being the case), then the principal mode will, through this nonlinearity, make questionable our entire ability to decompose the overall attenuation into a sum over frequencies. Stated differently, the quality factors corresponding to the weak harmonics will no longer be well defined physical parameters.

Here follows a quotation from Goldreich (1963):
*"... Darwin and Jeffreys both wrote the tide-raising potential as the sum of periodic potentials. They then proceeded to consider the response of the planet to each of the potentials separately. At first glance this might seem proper since the tidal strains are very small and should add linearly. The stumbling block in this procedure, however, is the amplitude dependence of the specific dissipation function. In the case of the Earth, it has been shown by direct measurement that $Q$ varies by an order of magnitude if we compare the tide of frequency $2\omega - 2n$ with the tides of frequencies $2\omega - n$, $2\omega - 3n$, and $\frac{3}{2}n$. This is because these latter tides have amplitudes which are smaller than the principle tide (of frequency $2\omega - 2n$ ) by a factor of eccentricity or about 0.05. It may still appear that we can allow for this amplitude dependence of Q merely by adopting an amplitude dependence for the phase lags of the different tides. Unfortunately, this is really not sufficient since a tide of small amplitude will have a phase lag which increases when its peak is reinforcing the peak of the tide of the major amplitude. This non-linear behaviour cannot be treated in detail since very little is known about the response of the planets to tidal forces, except for the Earth."*

On these grounds, Goldreich concluded the paragraph with an important warning that we *"use the language of linear tidal theory, but we must keep in mind that our numbers are really only parametric fits to a non-linear problem."*

In order to mark the line beyond which this caveat cannot be ignored, let us first of all recall that the linear approximation remains applicable insofar as the strains do not approach the nonlinearity threshold, which for most minerals is of order $10^{-6}$. On approach to that threshold, the quality factors may become dependent upon the strain magnitude. In other words, in an attempt to extend the expansion (1 - 2) to the nonlinear case, we shall have to introduce, instead of $Q(\chi_i)$, some new functions $Q(\chi_i\,,\;E_{peak}(\chi_i)\,,\;E_{overall})$ . (Another complication is that in the nonlinear regime new frequencies will be generated, but we shall not go there.) Now consider a superposition of two forcing stresses – one at the frequency $\chi_1$ and another at $\chi_2$ . Let the amplitude $E_{peak}(\chi_1)$ be close or above the nonlinearity threshold, and $E_{peak}(\chi_2)$ be by an order or two of magnitude smaller than $E_{peak}(\chi_1)$ . To adapt the linear machinery (1 - 2) to the nonlinear situation, we have to write it as

$$\langle\,\dot{E}\,\rangle\;=\;\langle\,\dot{E}_1\,\rangle\;+\;\langle\,\dot{E}_2\,\rangle\;=\;-\;\chi_1\;\frac{E_{peak}(\chi_1)}{Q(\,\chi_1\,,\;E_{peak}(\chi_1)\,)}\;-\;\chi_2\;\frac{E_{peak}(\chi_2)}{Q(\chi_2\,,\;E_{peak}(\chi_1)\,,\;E_{peak}(\chi_2)\,)}\quad, \tag{6}$$

the second quality factor bearing a dependence not only upon the frequency $\chi_2$ and the appropriate magnitude $E_{peak}(\chi_2)$ , but also upon the magnitude of the *first* mode, $E_{peak}(\chi_1)$ , – this happens because it is the first mode which makes a leading contribution into the overall stress. Even if (6) can be validated as an extension of (1 - 2) to nonlinear regimes, we should remember that the second term in (6) is much smaller than the first one (because we agreed that $E_{peak}(\chi_2)\ll E_{peak}(\chi_1)$ ). This results in two quandaries. The first one (not mentioned by Goldreich) is that a nonlinearity-caused non-smooth behaviour of $Q(\,\chi_1\,,\;E_{peak}(\chi_1)\,)$ will cause variations of the first term in (6), which may exceed or be comparable to the entire second term. The second one (mentioned in the afore quoted passage from Goldreich) is the phenomenon of nonlinear superposition, i.e., the fact that the smaller-amplitude tidal harmonic has a higher dissipation rate (and, therefore, a larger phase lag) whenever the peak of this harmonic is reinforcing the peak of the principal mode. Under all these circumstances, fitting experimental data to (6) will become a risky business. Specifically, it will become impossible to reliably measure the frequency dependence of the second quality factor; therefore the entire notion of the quality factor will, in regard to the second frequency, become badly defined.

We shall not dwell on this topic in quantitative detail, leaving it for a future work. The only mentioning it is to draw the readers' attention to the existing difficulty stemming from the shortcomings of the extension of (1 - 2) to nonlinear regimes. In what follows, we shall consider linear deformations only.

## 3 Darwin (1879) and Kaula (1964)

The potential produced at point $\vec{\boldsymbol{R}}=(R\,,\,\lambda\,,\,\phi)$ by a mass $M^*$ located at $\vec{\boldsymbol{r}}^{\,*}=(r^*,\,\lambda^*\,,\,\phi^*)$ is

$$W(\vec{\boldsymbol{R}}\,,\,\vec{\boldsymbol{r}}^{\,*})\;=\;-\,\frac{G\;M^*}{r^{\,*}}\sum_{l=2}^{\infty}\left(\frac{R}{r^{\,*}}\right)^{l}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(\sin\phi)P_{lm}(\sin\phi^*)\;\cos m(\lambda-\lambda^*)\quad. \tag{7}$$

When a tide-raising secondary located at $\vec{\boldsymbol{r}}^{\,*}$ distorts the shape of the primary, the potential generated by the primary at some exterior point $\vec{\boldsymbol{r}}$ gets changed. In the linear approximation, its variation is:

$$U(\vec{\boldsymbol{r}})\;=\;\sum_{l=2}^{\infty}\,k_l\,\left(\frac{R}{r}\right)^{l+1}\;W_l(\vec{\boldsymbol{R}}\,,\;\vec{\boldsymbol{r}}^{\,*})\quad, \tag{8}$$

$k_l$ being the $l$th Love number, $R$ now being the mean equatorial radius of the primary, $\vec{\boldsymbol{R}}\;=\;(R\,,\,\phi\,,\,\lambda)$ being a surface point, $\vec{\boldsymbol{r}}^{\,*}\;=\;(r^*\,,\,\phi^*\,,\,\lambda^*)$ being the coordinates of the tide-raising secondary, $\vec{\boldsymbol{r}}\;=\;(r\,,\,\phi\,,\,\lambda)$ being an exterior point located above the surface point $\vec{\boldsymbol{R}}$ at a radius $r\;\geq\;R$, and the longitudes being reckoned from a fixed meridian on the primary.

Substitution of (7) into (8) entails

$$U(\vec{\boldsymbol{r}}) \;=\; -\,G\,M^*\sum_{l=2}^{\infty}k_l\,\frac{R^{2l+1}}{r^{l+1}\,{r^*}^{l+1}}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})P_{lm}(\sin\phi)P_{lm}(\sin\phi^*)\;\cos m(\lambda-\lambda^*)\quad . \qquad (9)$$

A different expression for the tidal potential was offered by Kaula (1961, 1964), who developed a powerful technique that enabled him to switch from the spherical coordinates to the Kepler elements $(\,a^*,\,e^*,\,i^*,\,\Omega^*,\,\omega^*,\,\mathcal{M}^*\,)$ and $(\,a,\,e,\,i,\,\Omega,\,\omega,\,\mathcal{M}\,)$ of the secondaries located at $\vec{\boldsymbol{r}}^{\,*}$ and $\vec{\boldsymbol{r}}$. Application of this technique to (9) results in

$$U(\vec{\boldsymbol{r}}) \;=\; -\,\sum_{l=2}^{\infty}\,k_l\,\left(\frac{R}{a}\right)^{l+1}\,\frac{G\,M^*}{a^*}\,\left(\frac{R}{a^*}\right)^{l}\sum_{m=0}^{l}\,\frac{(l-m)!}{(l+m)!}\,(\,2 \qquad (10)$$

$$-\,\delta_{0m}\,)\sum_{p=0}^{l}F_{lmp}(i^*)\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)\;\cos\left[\left(v^*_{lmpq}-m\theta^*\right)-\left(v_{lmhj}-m\theta\right)\right]\quad ,$$

where

$$v^*_{lmpq}\;\equiv\;(l-2p)\omega^*\,+\,(l-2p+q)\mathcal{M}^*\,+\,m\,\Omega^*\quad , \qquad (11)$$

$$v_{lmhj}\;\equiv\;(l-2h)\omega\,+\,(l-2h+j)\mathcal{M}\,+\,m\,\Omega\quad , \qquad (12)$$

and $\theta\;=\;\theta^*$ is the sidereal angle.

While (10) and (9) are equivalent for an idealised elastic planet with an instant response of the shape, the situation becomes more involved when dissipation-caused delays come into play. Kaula's expression (10), as well as its truncated, Darwin's version,[1] is capable of accommodating separate phase lags for each harmonic involved:

$$U(\vec{\boldsymbol{r}}) \;=\; -\,\sum_{l=2}^{\infty}\,k_l\,\left(\frac{R}{a}\right)^{l+1}\,\frac{G\,M^*}{a^*}\,\left(\frac{R}{a^*}\right)^{l}\sum_{m=0}^{l}\,\frac{(l-m)!}{(l+m)!}\,(\,2\,- \qquad (13)$$

$$\delta_{0m}\,)\sum_{p=0}^{l}F_{lmp}(i^*)\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)\;\cos\left[\left(v^*_{lmpq}-m\theta^*\right)-\left(v_{lmhj}-m\theta\right)-\epsilon_{lmpq}\right]\quad .$$

where

$$\epsilon_{lmpq}=\left[\,(l-2p)\,\dot{\omega}^*\,+\,(l-2p+q)\,\dot{\mathcal{M}}^*\,+\,m\,(\dot{\Omega}^*\,-\,\dot{\theta}^*)\,\right]\,\Delta t_{lmpq}\,=\,\omega^*_{lmpq}\,\Delta t_{lmpq}\,=\,\pm\,\chi^*_{lmpq}\,\Delta t_{lmpq}\quad , \qquad (14)$$

is the phase lag interconnected with the quality factor via $Q_{lmpq}\;=\;\cot\,|\epsilon_{lmpq}|$ . The tidal harmonic $\omega^*_{lmpq}$ introduced in (14) is

$$\omega^*_{lmpq}\;\equiv\;(l-2p)\,\dot{\omega}^*\,+\,(l-2p+q)\,\dot{\mathcal{M}}^*\,+\,m\,(\dot{\Omega}^*\,-\,\dot{\theta}^*)\quad , \qquad (15)$$

[1] While the treatment by Kaula (1964) entails the infinite Fourier series (10), the developments by Darwin (1879) furnish its partial sum with $|l|,\,|q|,\,|j|\;\leq\;2$. For a simple introduction into Darwin's method see Ferraz-Mello, Rodríguez & Hussmann (2008).

while the positively-defined quantity

$$\chi^*_{lmpq} \;\equiv\; |\,\omega^*_{lmpq}\,| \;=\; |\,(l-2p)\,\dot{\omega}^* \;+\; (l-2p+q)\,\dot{\mathcal{M}}^* \;+\; m\,(\dot{\Omega}^* \;-\; \dot{\theta}^*)\;| \qquad (16)$$

is the actual physical $lmpq$ frequency excited by the tide in the primary. The corresponding positively-defined time delay $\Delta t_{lmpq}$ depends on this physical frequency, the functional forms of this dependence being different for different materials.

Formulae (10) and (13) constitute the principal result of Kaula's theory of tides. Most importantly, Kaula's formalism imposes no *a priori* constraint on the form of frequency-dependence of the lags.

# 4 The Darwin-Kaula-Goldreich expansion for the tidal torque

Now we are prepared to calculate the planet-perturbing tidal torque. Since in what follows we shall dwell on the low-inclination case, it will be sufficient to derive the torque's component orthogonal to the planetary equator:

$$\tau \;=\; -\,M\,\frac{\partial U(\vec{\boldsymbol{r}})}{\partial \theta}\quad , \qquad (17)$$

$M$ being the mass of the tide-disturbed satellite, and the "minus" sign emerging due to our choice not of the astronomical but of the physical sign convention. Adoption of the latter convention implies the emergence of a "minus" sign in the expression for the potential of a point mass: $-\,Gm/r$. This "minus" sign then shows up on the right-hand sides of (7), (9), (10), and (13). It is then compensated by the "minus" sign standing in (17).

The right way of calculating $\partial U(\vec{\boldsymbol{r}})/\partial\theta$ is to take the derivative of (13) with respect to $\theta$, and then[2] to get rid of the sidereal angle completely, by imposing the constraint $\theta^* \;=\; \theta$. This will yield:

$$\tau = -\sum_{l=2}^{\infty} k_l \left(\frac{R}{a}\right)^{l+1} \frac{G\,M^*\,M}{a^*}\left(\frac{R}{a^*}\right)^{l} \sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}2m\,\sum_{p=0}^{l}F_{lmp}(i^*)\sum_{q=-\infty}^{\infty}G_{lpq}(e^*)$$

$$\sum_{h=0}^{l}F_{lmh}(i)\sum_{j=-\infty}^{\infty}G_{lhj}(e)\;\sin\left[\,v^*_{lmpq}\;-\;v_{lmhj}\;-\;\epsilon_{lmpq}\,\right]\quad , \qquad (18)$$

In the case of the tide-raising satellite coinciding with the tide-perturbed one, all the elements become identical to their counterparts with an asterisk. For a primary body not in a tidal lock with its satellite,[3] it is sufficient to limit our consideration to the constant part of the torque,[4]

[2] Be mindful that our intention here is to differentiate not $\cos\left[\left(v^*_{lmpq}-m\theta^*\right)-\left(v_{lmhj}-m\theta\right)\right]$ but $\cos\left[\left(v^{*(delayed)}_{lmpq}-m\theta^{*(delayed)}\right)-\left(v_{lmhj}-m\theta\right)\right]$. Hence the said sequence of operations.

[3] With $\alpha$ denoting the librating angle, the locking condition reads: $\theta \;=\; \Omega+\omega+\mathcal{M}+180^o+\alpha+O(i^2)$. Insertion thereof into (15) results in: $\omega^*_{lmpq}\;\equiv\;(l-2p-m)\;\dot{\omega}^*\;+\;(l-2p+q-m)\;\dot{\mathcal{M}}^*$, where we have neglected $-m\dot{\alpha}$ on account of $\alpha$ being extremely small. Clearly, the indices can assume more than one set of values corresponding to one tidal frequency. This way, the case of libration is more involved than that of tidal despinning, and deserves a separate consideration.

[4] The tide-raising and tidally-perturbed satellites being the same body does *not* yet mean that the indices

a part for which the indices $(p\,,\,q)$ coincide with $(h\,,\,j)$, and therefore $v_{lmhj}$ cancels with $v^*_{lmpq}$. This will give us:

$$\tau \;=\; \sum_{l=2}^{\infty} 2\, k_l\; G\; M^2\;\; R^{2l+1}\, a^{-2l-2} \sum_{m=0}^{l} \frac{(l\,-\,m)!}{(l\,+\,m)!}\; m \sum_{p=0}^{l} F^2_{lmp}(i) \sum_{q=-\infty}^{\infty} G^2_{lpq}(e)\;\sin\epsilon_{lmpq} \quad . \tag{19}$$

The expression gets considerably simplified if we restrict ourselves to the case of $l\,=\,2$. Since $0\,\leq\,m\,\leq\,l$, and since $m$ enters the expansion as a multiplier, we see that only $m\,=\,1\,,\,2$ actually matter. As $0\,\leq\,p\,\leq\,l$, we are left with only six relevant $F$'s, those corresponding to $(lmp)\,=$ (210), (211), (212), (220), (221), and (222). By a direct inspection of the table of $F_{lmp}$ we find that five of these six functions happen to be $O(i)$ or $O(\,i^2\,)$, the sixth one being $F_{220}\,=\,\frac{3}{4}\,(\,1\,+\,\cos i\,)^2\,=\,3\,+\,O(i^2)$. Thus we obtain, in the leading order of $i$ :

$$\tau_{l=2} \;=\; \frac{3}{2} \sum_{q=-\infty}^{\infty} G\; M^2\;\; R^5\; a^{-6}\; G^2_{20q}(e)\; k_2\;\sin\epsilon_{220q} \;+\; O(i^2/Q) \quad . \tag{20}$$

The leading term of the expansion is

$$\tau_{2200} \;=\; \frac{3}{2}\; G\, M^2\; k_2\; R^5\;\; a^{-6} \sin\epsilon_{2200} \quad . \tag{21}$$

Switching from the lags to quality factors via formula[5]

$$Q_{lmpq} \;=\; |\;\cot\epsilon_{lmpq}\,| \quad , \tag{22}$$

we obtain:

$$\sin\epsilon_{lmpq} \;=\; \sin|\epsilon_{lmpq}|\;\;\mathrm{sgn}\,\omega_{lmpq} \;=\; \frac{\mathrm{sgn}\,\omega_{lmpq}}{\sqrt{1\;+\;\;\cot^2\epsilon_{lmpq}}} \;=\; \frac{\mathrm{sgn}\,\omega_{lmpq}}{\sqrt{1\;+\;Q^2_{lmpq}}} \;=\; \frac{\mathrm{sgn}\,\omega_{lmpq}}{Q_{lmpq}} + O(Q^{-3}) \quad , \tag{23}$$

whence

$$\tau_{l=2} \;=\; \frac{3}{2} \sum_{q=-\infty}^{\infty} G\; M^2\;\; R^5\; a^{-6}\; G^2_{20q}(e)\; k_2\; \frac{\mathrm{sgn}\,\omega_{220q}}{Q_{220q}} \;+\; O(i^2/Q) \;+\; O(Q^{-3}) \quad .$$

Now, let us simplify the sign multiplier. If in expression (15) for $\omega_{lmpq}$ we get rid of the

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$(p\,,\,q)$ coincide with with $(h\,,\,j)$. These are two independent sets of indices, wherewith the terms of two Fourier series are numbered, expression (18) being a product of those two series. This product contains a constant part, as well as short-period terms proportional to $\dot{\mathcal{M}}$ and long-period terms proportional to $\dot{\omega}$. The short-period terms get averaged out over a period of the tidal flexure, while the long-period terms get averaged out over longer times, *provided the periapse is precessing and not librating.* Expression (19) furnishes the constant part of the torque. Fortunately, this is sufficient for our further calculations.

[5] The phase lag $\epsilon_{lmpq}$ is introduced in (14), while the tidal harmonic $\omega_{lmpq}$ is given by (15). The quality factor $Q_{lmpq}\;=\;|\,\cot\epsilon_{lmpq}\,|$ is, for physical reasons, positively defined. Hence the multiplier $\mathrm{sgn}\,\omega_{lmpq}$ in (23). (As ever, the function $\mathrm{sgn}(x)$ is defined to assume the values $+1$, $-1$, or $0$ for positive, negative, or vanishing $x$, correspondingly.)

Mind that no factor of two appears in (22 - 23), because $\epsilon$ is a phase lag, not a geometric angle.

redundant asterisks, replace[6] $\dot{\mathcal{M}}$ with $\dot{\mathcal{M}}_0 + n \approx n$, and set $l = m = 2$ and $p = 0$, the outcome will be:

$$\mathrm{sgn}\,\omega_{220q} \;=\; \mathrm{sgn}\,\left[2\,\dot{\omega}\,+\,(2+q)\,n\,+\,2\,\dot{\Omega}-\,2\,\dot{\theta}\right] \;=\; \mathrm{sgn}\,\left[\dot{\omega}\,+\,\left(1\,+\,\frac{q}{2}\right)\,n\,+\,\dot{\Omega}-\,\dot{\theta}\right] \quad .$$

As the node and periapse precessions are slow, the above expression may be simplified to

$$\mathrm{sgn}\,\left[\left(1\,+\,\frac{q}{2}\right)\,n\,-\,\dot{\theta}\right] \quad .$$

All in all, the approximation for the torque assumes the form:

$$\tau_{l=2} \;=\; \frac{3}{2}\,\sum_{q=-\infty}^{\infty}\,G\,M^2\;R^5\,a^{-6}\,G^2_{20q}(e)\,k_2\,Q^{-1}_{220q}\;\;\mathrm{sgn}\,\left[\left(1\,+\,\frac{q}{2}\right)\,n\,-\,\dot{\theta}\right]+O(i^2/Q)+O(Q^{-3}) \quad . \qquad (24)$$

That the sign of the right-hand side in the above formula is correct can be checked through the following obvious observation: for a sufficiently high spin rate $\dot{\theta}$ of the planet, the multiplier $\mathrm{sgn}\,\left[\left(1\,+\,\frac{q}{2}\right)\,n\,-\,\dot{\theta}\right]$ becomes negative. Thereby the overall expression for $\tau_{l=2}$ acquires a "minus" sign, so that the torque points out in the direction of rotation opposite to the direction of increase of the sidereal angle $\theta$. This is exactly how it should be, because for a fixed $q$ and a sufficiently fast spin the $q$'s component of the tidal torque must be decelerating and driving the planet to synchronous rotation.

Expansion (24) was written down for the first time, without proof, by Goldreich & Peale (1966). A schematic proof was later offered by Dobrovolskis (2007).

# 5 Can the quality factor scale as a positive power of the tidal frequency?

As of now, the functional form of the dependence $Q(\chi)$ for Jovian planets remains unknown. For terrestrial planets, the model $Q\;\sim\;1/\chi$ is definitely incompatible with the geophysical data. A convincing volume of measurements firmly witnesses that $Q$ of the mantle scales as the tidal frequency to a *positive* fractional power:

$$Q \;=\; \mathcal{E}^{\alpha}\,\chi^{\alpha} \quad , \qquad \text{where} \quad \alpha\;=\;0.3\,\pm\,0.1 \quad , \qquad (25)$$

$\mathcal{E}$ being an integral rheological parameter with dimensions of time. This rheology is incompatible with the postulate of frequency-independent time-delay. Therefore an honest calculation should be based on averaging the Darwin-Kaula-Goldreich formula (24), with the actual scaling law (25) inserted therein, and with the appropriate dependence $\Delta t_{lmpq}(\,\chi_{lmpq}\,)$ taken into account.[7]

[6] While in the undisturbed two-body setting $\mathcal{M}\;=\;\mathcal{M}_0+n\,(t-t_0)$ and $\dot{\mathcal{M}}=n$, under perturbation these relations get altered. One possibility is to introduce (following Tisserand 1893) an *osculating mean motion* $n(t)\;\equiv\;\sqrt{\mu/a(t)^3}$, and to stick to this definition under perturbation. Then the mean anomaly will evolve as $\mathcal{M}\;=\;\mathcal{M}_0(t)+\int_{t_o}n(t)\,dt$, whence $\dot{\mathcal{M}}=\dot{\mathcal{M}}_0(t)+n(t)$.

Other possibilities include introducing an *apparent* mean motion, i.e., defining $n$ either as the mean-anomaly rate $d\mathcal{M}/dt$, or as the mean-longitude rate $dL/dt\;=\;d\Omega/dt\,+\,d\omega/dt\,+\,d\mathcal{M}/dt$ (as was done by Williams et al. 2001). While the first-order perturbations in $a(t)$ and in the osculating mean motion $\sqrt{\mu/a(t)^3}$ do not have secular rates, the epoch terms typically do have secular rates. Hence the difference between the apparent mean motion defined as $dL/dt$ (or as $d\mathcal{M}/dt$) and the osculating mean motion $\sqrt{\mu/a(t)^3}$. I am thankful to James G. Williams for drawing my attention to this circumstance (J.G. Williams, private communication).

[7] For the dependence of $\Delta t_{lmpq}$ upon $\chi_{lmpq}$ see Efroimsky & Lainey 2007.

## 5.1 The “paradox”

Although among geophysicists the scaling law (25) has long become common knowledge, in the astronomical community it is often met with prejudice. The prejudice stems from the fact that, in the expression for the torque, $Q$ stands in the denominator:

$$\tau \sim \frac{1}{Q} \quad . \tag{26}$$

At the instant of crossing the synchronous orbit, the principal tidal frequency $\chi_{2200}$ becomes nil, for which reason insertion of

$$Q \sim \chi^{\alpha} \quad , \qquad \alpha > 0 \tag{27}$$

into (26) seems to entail an infinitely large torque at the instant of crossing:

$$\tau \sim \frac{1}{Q} \sim \frac{1}{\chi^{\alpha}} \rightarrow \infty \quad , \qquad \text{for} \quad \chi \rightarrow 0 \quad , \tag{28}$$

a clearly unphysical result.

Another, very similar objection to (25) originates from the fact that the quality factor is inversely proportional to the phase shift: $Q \sim 1/\epsilon$. As the shift (14) vanishes on crossing the synchronous orbit, one may think that the value of the quality factor must, effectively, approach infinity. On the other hand, the principal tidal frequency vanishes on crossing the synchronous orbit, for which reason (25) makes the quality factor vanish. Thus we come to a contradiction.

For these reasons, the long-entrenched opinion is that these models introduce discontinuities into the expression for the torque, and can thus be considered as unrealistic.

It is indeed true that, while law (25) works over scales shorter than the Maxwell time (about $10^2$ yr for most minerals), it remains subject to discussion in regard to longer timescales. Nonetheless, it should be clearly emphasised that the infinities emerging at the synchronous-orbit crossing can in no way disprove any kind of rheological model. They can only disprove the flawed mathematics whence they provene.

## 5.2 A case for reasonable doubt

To evaluate the physical merit of the alleged infinite-torque “paradox”, recall the definition of the quality factor. As part and parcel of the linearity approximation, the overall damping inside a body is expanded in a sum of attenuation rates corresponding to each periodic disturbance:

$$\langle \dot{E} \rangle = \sum_i \langle \dot{E}(\chi_i) \rangle \tag{29}$$

where, at each frequency $\chi_i$ ,

$$\langle \dot{E}(\chi_i) \rangle = - 2 \chi_i \frac{\langle E(\chi_i) \rangle}{Q(\chi_i)} = - \chi_i \frac{E_{peak}(\chi_i)}{Q(\chi_i)} \quad , \tag{30}$$

$\langle \ldots \rangle$ designating an average over a flexure cycle, $E(\chi_i)$ denoting the energy of deformation at the frequency $\chi_i$ , and $Q(\chi_i)$ being the quality factor of the medium at this frequency.

This definition by itself leaves enough room for doubt in the above "paradox". As can be seen from (30), the dissipation rate is proportional not to $1/Q(\chi)$ but to $\chi/Q(\chi)$ . This way, for the dependence $Q \sim \chi^{\alpha}$, the dissipation rate $\langle \dot{E} \rangle$ will behave as $\chi^{1-\alpha}$ . In the limit of $\chi \to 0$, this scaling law portends no visible difficulties, at least for the values of $\alpha$ up to unity. While raising $\alpha$ above unity may indeed be problematic, there seem to be no fundamental obstacle to having materials with positive $\alpha$ taking values up to unity. So far, such values of $\alpha$ have caused no paradoxes, and there seems to be no reason for any infinities to show up.

## 5.3 The phase shift and the quality factor

As another preparatory step, we recall that, rigorously speaking, the torque is proportional not to the phase shift $\epsilon$ itself but to $\sin\epsilon$. From (23) and (25) we obtain:

$$|\sin\epsilon| \;=\; \frac{1}{\sqrt{1\,+\,Q^2}} \;=\; \frac{1}{\sqrt{1\,+\,\mathcal{E}^{2\alpha}\,\chi^{2\alpha}}} \quad . \tag{31}$$

We see that only for large values of $Q$ one can approximate $|\sin\epsilon|$ with $1/Q$ (crossing of the synchronous orbit *not* being the case). Generally, in any expression for the torque, the factor $1/Q$ must always be replaced with $1/\sqrt{1\,+\,Q^2}$ . Thus instead of (26) we must write:

$$\tau \;\sim\; |\sin\epsilon| \;=\; \frac{1}{\sqrt{1\,+\,Q^2}} \;=\; \frac{1}{\sqrt{1\,+\,\mathcal{E}^{2\alpha}\,\chi^{2\alpha}}} \quad , \tag{32}$$

$\mathcal{E}$ being a dimensional constant from (25).

Though this immediately spares us from the fake infinities at $\chi \to 0$, we still are facing this strange situation: it follows from (31) that, for a positive $\alpha$ and vanishing $\chi$, the phase lag $\epsilon$ must be approaching $\pi/2$, thereby inflating the torque to its maximal value (while on physical grounds the torque should vanish for zero $\chi$). Evidently, some important details are still missing from the picture.

## 5.4 The stone rejected by the builders

To find the missing link, recall that Kaula (1964) described tidal damping by employing the method suggested by Darwin (1880): he accounted for attenuation by merely adding a phase shift to every harmonic involved – an empirical approach intended to make up for the lack of a consistent hydrodynamical treatment with viscosity included. It should be said, however, that prior to the work of 1880 Darwin had published a less known article (Darwin 1879), in which he attempted to construct a self-consistent theory, one based on the viscosity factor of the mantle, and not on empirical phase shifts inserted by hand. Darwin's conclusions of 1879 were summarised and explained in a more general mathematical setting by Alexander (1973).

The pivotal result of the self-consistent hydrodynamical study is the following. When a variation of the potential of a tidally disturbed planet, $U(\vec{\boldsymbol{r}})$, is expanded over the Legendre functions $P_{lm}(\sin\phi)$, each term of this expansion will acquire not only a phase lag but also a factor describing a change in amplitude. This forgotten factor, derived by Darwin (1879), is nothing else but $\cos\epsilon$ . Its emergence should in no way be surprising if we recall that the damped, forced harmonic oscillator

$$\ddot{x} \,+\, 2\,\gamma\,\dot{x} \,+\, \omega_o^2\,x \;=\; F\,e^{i\,\lambda\,t} \tag{33}$$

evolves as

$$x(t) \;=\; C_1\; e^{(-\gamma + i\sqrt{\omega_o^2 - \lambda^2}\,)\,t} \;+\; C_2\; e^{(-\gamma + i\sqrt{\omega_o^2 - \lambda^2}\,)\;t} \;+\; \frac{F\;\cos\epsilon}{\omega_o^2\,-\,\lambda^2}\; e^{i\,(\lambda\,t\,-\,\epsilon)} \quad , \tag{34}$$

where the phase lag is

$$\tan\epsilon \;=\; 2\,\gamma\,\lambda\,\left(\omega_o^2\,-\,\lambda^2\right) \quad , \tag{35}$$

and the first two terms in (34) are damped away in time.[8]

In the works by Darwin's successors, the allegedly irrelevant factor of $\cos\epsilon$ fell through the cracks, because the lag was always asserted to be small. In reality, though, each term in the Fourier expansions (13), (18 - 21), and (24) should be amended with $\cos\epsilon_{lmpq}$ . For the same reason, instead of (32), we should write down:

$$\tau\;\sim\;|\,\cos\epsilon\;\;\sin\epsilon\,|\;=\;\frac{Q}{\sqrt{1\,+\,Q^2}}\;\frac{1}{\sqrt{1\,+\,Q^2}}\;=\;\frac{\mathcal{E}^{\alpha}\;\chi^{\alpha}}{1\,+\,\mathcal{E}^{2\alpha}\;\chi^{2\alpha}} \quad , \tag{36}$$

At this point, it would be tempting to conclude that, since (71) vanishes in the limit of $\chi\rightarrow 0$ , *for any sign of* $\alpha$ , then no paradoxes happens on the satellite's crossing the synchronous orbit. Sadly, this straightforward logic would be too simplistic.

In fact, prior to saying that $\cos\epsilon\,\sin\epsilon\,\rightarrow\,0$, we must take into consideration one more subtlety missed so far. As demonstrated in the Appendix, taking the limit of $Q\,\rightarrow\,0$ is a nontrivial procedure, because at small values of $Q$ the interconnection between the lag and the Q factor becomes very different from the conventional $Q=\cot|\epsilon|$. A laborious calculation shows that, for $Q<1-\pi/4$ , the relation becomes:

$$\sin\epsilon\,\cos\epsilon\;=\;\pm\,(3Q)^{1/3}\,\left[1-\frac{4}{5}(3Q)^{2/3}+O(Q^{4/3})\right] \quad ,$$

which indeed vanishes for $Q\,\rightarrow\,0$. Both $\epsilon_{2200}$ and the appropriate component of the torque change their sign on the satellite crossing the synchronous orbit.

So the main conclusion remains in force: nothing wrong happens on crossing the synchronous orbit, Q.E.D.

# 6 Conclusions

In the article thus far we have punctiliously spelled out some assumptions that often remain implicit, and brought to light those steps in calculations, which are often omitted as

[8] As demonstrated by Alexander (1973), this example indeed has relevance to the hydrodynamical theory of Darwin, and is not a mere illustration. Alexander (1973) also explained that the emergence of the $\cos\epsilon$ factor is generic. (Darwin (1879) had obtained it in the simple case of $l\;=\;2$ and for a special value of the Love number: $k_l\,=\,1.5$ .)

A further investigation of this issue was undertaken in a comprehensive work by Churkin (1998), which unfortunately has never been published in English because of a tragic death of its Author. In this preprint, Churkin explored the frequency-dependence of both the Love number $k_2$ and the quality factor within a broad variety of rheological models, including those of Maxwell and Voight. It follows from Churkin's formulae that within the Voight model the dynamical $k_2$ relates to the static one as $\cos\epsilon$ . In the Maxwell and other models, the ratio approaches $\cos\epsilon$ in the low-frequency limit.

"self-evident". This has helped us to explain that no "paradoxes" ensue from the frequency-dependence $Q \sim \chi^{\alpha}$ , $\alpha = 0.3 \pm 0.1$ , which is in fact the actual dependence found for the mantle and crust.

This preprint is a pilot paper. A more comprehensive treatise on tidal torques is to be published. (Efroimsky & Williams 2009)

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## Appendix.

### The lag and the quality factor: is the formula $Q = \cot|\epsilon|$ universal?

The interrelation between the quality factor $Q$ and the phase lag $\epsilon$ is long-known to be

$$Q \, = \, \cot|\epsilon| \quad , \tag{37}$$

and its derivation can be found in many books. In Appendix A2 of Efroimsky & Lainey(2007), that derivation is reproduced, with several details that are normally omitted in the literature. Among other things, we pointed out that the interrelation has exactly the form (37) only in the limit of small lags. For large phase lags, the form of this relation will change considerably.

Since in section 9 of the current paper we address the case of large lags, it would be worth reconsidering the derivation presented in Efroimsky & Lainey (2007), and correcting a subtle omission made there. Before writing formulae, let us recall that, at each frequency $\chi$ in the spectrum of the deformation, the quality factor (divided by $2\pi$) is defined as the peak energy stored in the system divided by the energy damped over a cycle of flexure:

$$Q(\chi) \, \equiv \, - \, \frac{2\,\pi\,E_{peak}(\chi)}{\Delta E_{cycle}(\chi)} \quad , \tag{38}$$

where $\Delta E_{cycle}(\chi) \, < \, 0$ as we are talking about energy losses.[9]

An attempt to consider large lags (all the way up to $|\epsilon| \, = \, \pi/2$) sets the values of $Q/2\pi$ below unity. As the dissipated energy cannot exceed the energy stored in a free oscillator, the question becomes whether the values of $Q/2\pi$ can be that small. To understand that they can, recall that in this situation we are considering an oscillator, which is not free but is driven (and is overdamped). The quality factor being much less than unity simply implies that the

[9] We are considering flexure in the linear approximation. Thus at each frequency $\chi$ the appropriate energy loss over a cycle, $\Delta E_{cycle}(\chi)$, depends solely on the maximal energy stored at that same frequency, $E_{peak}(\chi)$.

eigenfrequencies get damped away during less than one oscillation. Nonetheless, motion goes on due to the driving force.

Now let us switch to the specific context of tides. To begin with, let us recall that the dissipation rate in a tidally distorted primary is well approximated by the work that the secondary carries out to deform the primary:

$$\dot{E} \; = \; - \int \rho \; \vec{\boldsymbol{V}} \; \cdot \; \nabla W \; d^3x \tag{39}$$

$\rho$ , $\vec{\boldsymbol{V}}$ , and $W$ denoting the density, velocity, and tidal potential inside the primary. The expression on the right-hand side can be transformed by means of the formula

$$\rho \, \vec{\boldsymbol{V}} \cdot \nabla W \; = \; \nabla \cdot (\rho \, \vec{\boldsymbol{V}} \, W) \; - \; W \; \vec{\boldsymbol{V}} \cdot \nabla \rho \; - \; W \, \nabla \cdot (\rho \, \vec{\boldsymbol{V}}) \; = \; \nabla \cdot (\rho \, \vec{\boldsymbol{V}} \, W) \; - \; W \; \vec{\boldsymbol{V}} \cdot \nabla \rho \; + \; W \, \frac{\partial \rho}{\partial t} \quad , \tag{40}$$

where the $W \; \vec{\boldsymbol{V}} \cdot \nabla \rho$ and $\partial \rho / \partial t$ terms may be omitted under the assumption that the primary is homogeneous and incompressible. In this approximation, the attenuation rate becomes simply

$$\dot{E} \; = \; - \int \nabla \; \cdot \; (\rho \; \vec{\boldsymbol{V}} \; W) \, d^3x \; = \; - \int \rho \, W \; \vec{\boldsymbol{V}} \; \cdot \; \vec{\mathbf{n}} \; dA \quad , \tag{41}$$

$\vec{\mathbf{n}}$ being the outward normal to the surface of the primary, and $dA$ being an element of the surface area. It is now clear that, under the said assertions, it is sufficient to take into account only the radial elevation rate, not the horizontal distortion. This way, formula (39), in application to a unit mass, will get simplified to

$$\dot{E} \; = \; \left( - \, \frac{\partial W}{\partial r} \right) \; \vec{\boldsymbol{V}} \cdot \vec{\mathbf{n}} \; = \; \left( - \, \frac{\partial W}{\partial r} \right) \frac{d\zeta}{dt} \quad , \tag{42}$$

$\zeta$ standing for the vertical displacement (which is, of course, delayed in time, compared to $W$ ). The amount of energy dissipated over a time interval $(t_o \, , \; t)$ is then

$$\Delta E \; = \; \int_{t_o}^{t} \left( - \, \frac{\partial W}{\partial r} \right) \; d\zeta \quad . \tag{43}$$

We shall consider the simple case of an equatorial moon on a circular orbit. At each point of the planet, the variable part of the tidal potential produced by this moon will read

$$W \; = \; W_o \; \cos \chi t \quad , \tag{44}$$

the tidal frequency being given by

$$\chi \; = \; 2 \; |n \; - \; \omega_p| \quad . \tag{45}$$

Let $\mathrm{g}$ denote the surface free-fall acceleration. An element of the planet's surface lying beneath the satellite's trajectory will then experience a vertical elevation of

$$\zeta \; = \; h_2 \, \frac{W_o}{\mathrm{g}} \; \cos(\chi t \; - \; |\epsilon|) \quad , \tag{46}$$

$h_2$ being the corresponding Love number, and $|\epsilon|$ being the *positive*[10] phase lag, which for the principal tidal frequency is simply the double geometric angle $\delta$ subtended at the primary's centre between the directions to the secondary and to the main bulge:

$$|\epsilon| \;=\; 2\,\delta \quad . \tag{47}$$

Accordingly, the vertical velocity of this element of the planet's surface will amount to

$$u \;=\; \dot{\zeta} \;=\; -\,h_2\,\chi\,\frac{W_o}{\mathrm{g}}\,\sin(\chi t\;-\;|\epsilon|) \;=\; -\,h_2\,\chi\,\frac{W_o}{\mathrm{g}}\,\left(\sin\chi t\,\cos|\epsilon|\;-\;\cos\chi t\,\sin|\epsilon|\right) \quad . \tag{48}$$

The expression for the velocity has such a simple form because in this case the instantaneous frequency $\chi$ is constant. The satellite generates two bulges – on the facing and opposite sides of the planet – so each point of the surface is uplifted twice through a cycle. This entails the factor of two in the expression (45) for the frequency. The phase in (47), too, is doubled, though the necessity of this is less evident, – see footnote 4 in Appendix A1 to Efroimsky & Lainey (2007).

The energy dissipated over a time cycle $T\;=\;2\pi/\chi$, per unit mass, will, in neglect of horizontal displacements, be

$$\Delta E_{_{cycle}} \;=\; \int_0^T u\left(-\,\frac{\partial W}{\partial r}\right)dt = -\left(-\,h_2\,\chi\frac{W_o}{\mathrm{g}}\right)\frac{\partial W_o}{\partial r}\int_{t=0}^{t=T}\cos\chi t\,\left(\sin\chi t\,\cos|\epsilon|\;-\;\cos\chi t\,\sin|\epsilon|\right)dt$$

$$=\;-\,h_2\,\chi\,\frac{W_o}{\mathrm{g}}\,\frac{\partial W_o}{\partial r}\,\sin|\epsilon|\,\frac{1}{\chi}\int_{\chi t\,=\,0}^{\chi t\,=\,2\pi}\cos^2\chi t\;\;d(\chi t)\;=\;-\,h_2\,\frac{W_o}{\mathrm{g}}\,\frac{\partial W_o}{\partial r}\,\pi\,\sin|\epsilon|\quad , \tag{49}$$

while the peak work carried out on the system during the cycle will read:

$$E_{_{peak}} \;=\; \int_{|\epsilon|/\chi}^{T/4} u\left(-\,\frac{\partial W}{\partial r}\right)dt = -\left(-\,h_2\,\chi\frac{W_o}{\mathrm{g}}\right)\frac{\partial W_o}{\partial r}\int_{t=|\epsilon|/\chi}^{t=T/4}\cos\chi t\,\left(\sin\chi t\,\cos|\epsilon|\;-\;\cos\chi t\,\sin|\epsilon|\right)dt$$

$$=\;\chi\,h_2\,\frac{W_o}{\mathrm{g}}\,\frac{\partial W_o}{\partial r}\left[\frac{\cos|\epsilon|}{\chi}\int_{\chi t\,=\,|\epsilon|}^{\chi t\,=\,\pi/2}\cos\chi t\;\sin\chi t\;\;d(\chi t)\;-\;\frac{\sin|\epsilon|}{\chi}\int_{\chi t\,=\,|\epsilon|}^{\chi t\,=\,\pi/2}\cos^2\chi t\;\;d(\chi t)\right]\quad . \tag{50}$$

In the appropriate expression in Appendix A1 to Efroimsky & Lainey (2007), the lower limit of integration was erroneously set to be zero. To understand that in reality integration over $\chi\,t$ should begin from $|\epsilon|$, one should superimpose the plots of the two functions involved, $\cos\chi t$ and $\sin(\chi t\;-\;|\epsilon|)$. The maximal energy gets stored in the system after integration through the entire interval over which both functions have the same sign. Hence $\chi t = |\epsilon|$ as the lower limit.

[10] Were we not considering the simple case of a circular orbit, then, rigorously speaking, the expression for $W$ would read not as $W_o\,\cos\chi t$ but as $W_o\,\cos\omega_{tidal}t$, the tidal frequency $\omega_{tidal}$ taking both positive and negative values, and the physical frequency of flexure being $\chi\,\equiv\,|\omega_{tidal}|$. Accordingly, the expression for $\zeta$ would contain not $\cos(\chi t\,-\,|\epsilon|)$ but $\cos(\omega_{tidal}t\,-\,\epsilon)$. As we saw in equation (24), the sign of $\epsilon$ is always the same as that of $\omega_{tidal}$. For this reason, one may simply deal with the physical frequency $\chi\,\equiv\,|\omega_{tidal}|$ and with the absolute value of the phase lag, $|\epsilon|$.

Evaluation of the integrals entails:

$$E_{peak} \;=\; h_2\,\frac{W_o}{\mathrm{g}}\,\frac{\partial W_o}{\partial r}\;\left[\,\frac{1}{2}\,\cos|\epsilon|\;-\;\frac{1}{2}\,\left(\,\frac{\pi}{2}\;-\;|\epsilon|\,\right)\,\sin|\epsilon|\,\right]$$

whence

$$Q^{-1}\;=\;\frac{-\,\Delta E_{cycle}}{2\,\pi\,E_{peak}}\;=\;\frac{1}{2\,\pi}\;\frac{\pi\;\sin|\epsilon|}{\frac{1}{2}\,\cos|\epsilon|\;-\;\frac{1}{2}\,\left(\,\frac{\pi}{2}\;-\;|\epsilon|\,\right)\,\sin|\epsilon|}\;=\;\frac{\tan|\epsilon|}{1\;-\;\left(\,\frac{\pi}{2}\;-\;|\epsilon|\,\right)\,\tan|\epsilon|}\quad. \tag{52}$$

As can be seen from (52), both the product $\sin\epsilon\,\cos\epsilon$ and the appropriate component of the torque attain their maxima when $Q\;=\;1\;-\;\pi/4\,$.

Usually, $|\epsilon|$ is small, and we arrive at the customary expression

$$Q^{-1}\;=\;\tan|\epsilon|\;+\;O(\epsilon^2)\quad. \tag{53}$$

In the opposite situation, when $Q\rightarrow 0$ and $|\epsilon|\rightarrow\pi/2\,$, it is convenient to consider the small difference

$$\xi\;\equiv\;\frac{\pi}{2}\;-\;|\epsilon|\quad, \tag{54}$$

in terms whereof the inverse quality factor will read:

$$Q^{-1}\;=\;\frac{\cot\xi}{1\;-\;\xi\;\cot\xi}\;=\;\frac{1}{\tan\xi\;-\;\xi}\;=\;\frac{1}{z\;-\;\arctan z}\;=\;\frac{1}{\frac{1}{3}\,z^3\,\left[1\;-\;\frac{3}{5}\,z^2\;+\;O(z^4)\right]}\quad, \tag{55}$$

where $z\;\equiv\;\tan\xi$ and, accordingly, $\xi\;=\;\arctan z\;=\;z\,-\,\frac{1}{3}\,z^3\;+\;\frac{1}{5}\,z^5\;+\;O(z^7)\;$. Formula (55) may, of course, be rewritten as

$$z^3\,\left[1\;-\;\frac{3}{5}\,z^2\;+\;O(z^4)\right]\;=\;3\,Q \tag{56}$$

or, the same, as

$$z\;=\;(3\,Q)^{1/3}\,\left[1\;+\;\frac{1}{5}\,z^2\;+\;O(z^4)\right]\quad. \tag{57}$$

While the zeroth approximation is simply $z\;=\;(3Q)^{1/3}\;+\;O(Q)$ , the first iteration gives:

$$\tan\xi\;\equiv\;z\;=\;(3Q)^{1/3}\,\left[1\;+\;\frac{1}{5}\,(3Q)^{2/3}\;+\;O(Q^{4/3})\right]\;=\;q\;\left[1\;+\;\frac{1}{5}\,q^2\;+\;O(q^4)\right]\quad, \tag{58}$$

with $q\;=\;(3Q)^{1/3}$ playing the role of a small parameter.

We now see that the customary relation (53) should be substituted, for large lags, i.e., for small[11] values of $Q$, with:

$$\tan|\epsilon|\;=\;(3Q)^{-1/3}\,\left[1\;-\;\frac{1}{5}\,(3Q)^{2/3}\;+\;O(Q^{4/3})\right] \tag{59}$$

[11] The afore-employed expansion of $\arctan z$ is valid for $|z|\,<\,1$. This inequality, along with (55), entails: $Q\;=\;z\;-\;\arctan z\;<\;1\;-\;\pi/4\,$.

The formula for the tidal torque contains a multiplier $\sin\epsilon\,\cos\epsilon$, whose absolute value can, for our purposes, be written down as

$$\sin|\epsilon|\,\cos|\epsilon| = \cos\xi\,\sin\xi = \frac{\tan\xi}{1+\tan^2\xi} = \frac{q\left[1+\frac{1}{5}\,q^2+O(q^4)\right]}{1+q^2\left[1+O(q^2)\right]} = (3Q)^{1/3}\left[1-\frac{4}{5}(3Q)^{2/3}+O(Q^{4/3})\right]\;, \quad (60)$$

whence

$$\sin\epsilon\;\cos\epsilon \;=\; \pm\,(3Q)^{1/3}\left[1-\frac{4}{5}(3Q)^{2/3}+O(Q^{4/3})\right]\;, \quad (61)$$

an expression vanishing for $Q\;\rightarrow\;0$ . Be mindful that both $\epsilon_{2200}$ and the appropriate component of the torque change their sign on the satellite crossing the synchronous orbit.

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