1. A Calculation of the Integral I_v in Memo TM97-01.

Start with a clean slate.

```
> restart;
```

*diffeqn: [depvar, diffcoeffs, difforder, diffvars, indvarchange, isdiff, MakeCoeffs, *
maxPDEorder, Reconstruct, type/diff, varchange

*utils: [annular_average, applyfunc, CombineTrig, CombTrigOpt, common_factor, *
*cosfix, csquare, debug_print, expansion, expansion2, getsqrts, is_real_parts, funcops, *
*FunctionCalls, polytest, pullout, rootfunc, RotAxis, signsqrt, small_divisors, *
termfunc, tmsg, topsqrt, topsqrts

1.1. Assumptions and Definitions.

Make some reasonable assumptions on the variables we'll be using, in order to help out the integrator as much as possible. We'll use g instead of γ, since γ in Maple is a reserved numerical value.

```
assume( S > 0, g >= 0, g < 1, p,real, lambda > 0,
        k > 0, phi,real, f,real );
```

```
about(S,g,p,lambda,k,phi,f);
Originally S, renamed S~:
  is assumed to be: RealRange(Open(0),infinity)

Originally g, renamed g~:
  is assumed to be: RealRange(0,Open(1))

Originally p, renamed p~:
  is assumed to be: real

Originally lambda, renamed lambda~:
  is assumed to be: RealRange(Open(0),infinity)

Originally k, renamed k~:
  is assumed to be: RealRange(Open(0),infinity)

Originally phi, renamed phi~:
  is assumed to be: real

Originally f, renamed f~:
  is assumed to be: real
```

```
Js := ( sin(Pi/lambda*p*S) - sin(Pi/lambda*p*S*g) )/(Pi*p/lambda);
```

$$Js := \frac{\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right) - \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p}$$

```
dJs := diff(Js,p);
```

$$dJs := \frac{\left(\frac{\cos\left(\frac{\pi p S}{\lambda}\right)\pi S}{\lambda} - \frac{\cos\left(\frac{\pi p S g}{\lambda}\right)\pi S g}{\lambda}\right)\lambda}{\pi p} - \frac{\left(\sin\left(\frac{\pi p S}{\lambda}\right) - \sin\left(\frac{\pi p S g}{\lambda}\right)\right)\lambda}{\pi p^2}$$

```
dJs := termfunc( dJs, factor );
```

$$dJs := -\frac{S\left(-\cos\left(\frac{\pi p S}{\lambda}\right) + \cos\left(\frac{\pi p S g}{\lambda}\right) g\right)}{p} - \frac{\left(\sin\left(\frac{\pi p S}{\lambda}\right) - \sin\left(\frac{\pi p S g}{\lambda}\right)\right)\lambda}{\pi p^2}$$

```
integrand := Js*dJs*sin(k*p+phi);
```

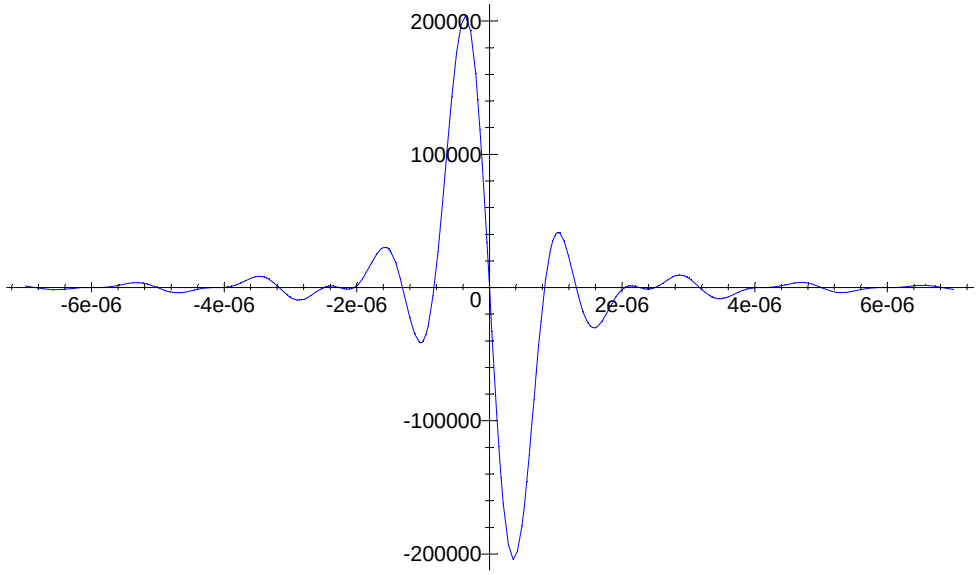
$$integrand := \left(\sin\left(\frac{\pi p S}{\lambda}\right) - \sin\left(\frac{\pi p S g}{\lambda}\right)\right)\lambda \left(-\frac{S\left(-\cos\left(\frac{\pi p S}{\lambda}\right) + \cos\left(\frac{\pi p S g}{\lambda}\right) g\right)}{p} - \frac{\left(\sin\left(\frac{\pi p S}{\lambda}\right) - \sin\left(\frac{\pi p S g}{\lambda}\right)\right)\lambda}{\pi p^2}\right) \sin(k p + \phi) / (\pi p)$$

Let's get some idea what this looks like.

```
F := fn(integrand,g,phi,lambda,k,S,p);
```

$$F := (g, \phi, \lambda, k, S, p) \rightarrow \left(\sin\left(\frac{\pi p S}{\lambda\sim}\right) - \sin\left(\frac{\pi p S g}{\lambda\sim}\right)\right)\lambda\sim \left(-\frac{S\left(-\cos\left(\frac{\pi p S}{\lambda\sim}\right) + \cos\left(\frac{\pi p S g}{\lambda\sim}\right) g\right)}{p} - \frac{\left(\sin\left(\frac{\pi p S}{\lambda\sim}\right) - \sin\left(\frac{\pi p S g}{\lambda\sim}\right)\right)\lambda\sim}{\pi p^2}\right) \sin(k p + \phi\sim) / (\pi p)$$

```
dp := 0.000007:
plot( F( 0.1, Pi/4, 5500*10^(-10), 0.1, 0.6, x ), x=-dp..dp,
      color=blue, numpoints=100 );
```



1.2. Perform the Definite Integration.

```
Iv := int( integrand, p=-infinity..infinity );
```

$$
\begin{aligned}
Iv := -\frac{1}{8}\lambda\Big(&\operatorname{signum}(\%2)\,\pi^2\cos(\phi)\,S\,g\,k-(-1+\operatorname{signum}(\%2))\,\pi^2\sin(\phi)\,S\,g\,k\\
&+(-1+\%4)\,\pi^2\sin(\phi)\,S\,k-(-1+\operatorname{signum}(\%3))\,\pi^2\sin(\phi)\,S\,g\,k-2\,\pi\cos(\phi)\,k^2\,\lambda\\
&+\operatorname{signum}(\%1)\,\pi\cos(\phi)\,k^2\,\lambda-\operatorname{signum}(\%2)\,\pi\cos(\phi)\,k^2\,\lambda-\operatorname{signum}(\%1)\,\pi^2\cos(\phi)\,S\,g\,k\\
&-(-1+\operatorname{signum}(\%3))\,\pi\sin(\phi)\,k^2\,\lambda+\frac{1}{2}\%5\,\pi\cos(\phi)\,k^2\,\lambda-\frac{1}{2}(-1+\%5)\,\pi\sin(\phi)\,k^2\,\lambda\\
&-\frac{1}{2}(-1+\%4)\,\pi\sin(\phi)\,k^2\,\lambda+\operatorname{signum}(\%3)\,\pi^2\cos(\phi)\,S\,g\,k\\
&+(-1+\operatorname{signum}(\%3))\,\pi^2\sin(\phi)\,S\,k-\%4\,\pi^2\cos(\phi)\,S\,k-(-1+\operatorname{signum}(\%1))\,\pi^2\sin(\phi)\,S\,k\\
&+\frac{1}{2}\%4\,\pi\cos(\phi)\,k^2\,\lambda+\operatorname{signum}(\%1)\,\pi^2\cos(\phi)\,S\,k-(-1+\operatorname{signum}(\%2))\,\pi^2\sin(\phi)\,S\,k\\
&+\operatorname{signum}(\%2)\,\pi^2\cos(\phi)\,S\,k+(-1+\%5)\,\pi^2\sin(\phi)\,S\,g\,k+(-1+\operatorname{signum}(\%2))\,\pi\sin(\phi)\,k^2\,\lambda\\
&-(-1+\operatorname{signum}(\%1))\,\pi\sin(\phi)\,k^2\,\lambda-\%5\,\pi^2\cos(\phi)\,S\,g\,k+\operatorname{signum}(\%3)\,\pi\cos(\phi)\,k^2\,\lambda\\
&-\operatorname{signum}(\%3)\,\pi^2\cos(\phi)\,S\,k+(-1+\operatorname{signum}(\%1))\,\pi^2\sin(\phi)\,S\,g\,k\Big)\Big/\pi^2+\frac{1}{8}\lambda\Big(\\
&(1+\operatorname{signum}(\%3))\,\pi\sin(\phi)\,k^2\,\lambda-(1+\operatorname{signum}(\%3))\,\pi^2\sin(\phi)\,S\,k-4\,I\,\pi\sin(\phi)\,k^2\,\lambda\\
&-\operatorname{signum}(\%2)\,\pi^2\cos(\phi)\,S\,g\,k+2\,\pi\cos(\phi)\,k^2\,\lambda-\operatorname{signum}(\%1)\,\pi\cos(\phi)\,k^2\,\lambda
\end{aligned}
$$

$$- (1 + \%5)\, \pi^2 \sin(\phi)\, S\, g\, k - (1 + \mathrm{signum}(\%1))\, \pi^2 \sin(\phi)\, S\, g\, k + \mathrm{signum}(\%2)\, \pi \cos(\phi)\, k^2 \lambda$$

$$+ \mathrm{signum}(\%1)\, \pi^2 \cos(\phi)\, S\, g\, k + \frac{1}{2} (1 + \%5)\, \pi \sin(\phi)\, k^2 \lambda + (1 + \mathrm{signum}(\%2))\, \pi^2 \sin(\phi)\, S\, g\, k$$

$$- (1 + \mathrm{signum}(\%2))\, \pi \sin(\phi)\, k^2 \lambda - \frac{1}{2} \%5\, \pi \cos(\phi)\, k^2 \lambda + (1 + \mathrm{signum}(\%2))\, \pi^2 \sin(\phi)\, S\, k$$

$$- \mathrm{signum}(\%3)\, \pi^2 \cos(\phi)\, S\, g\, k + \%4\, \pi^2 \cos(\phi)\, S\, k - \frac{1}{2} \%4\, \pi \cos(\phi)\, k^2 \lambda$$

$$+ (1 + \mathrm{signum}(\%1))\, \pi \sin(\phi)\, k^2 \lambda - \mathrm{signum}(\%1)\, \pi^2 \cos(\phi)\, S\, k - \mathrm{signum}(\%2)\, \pi^2 \cos(\phi)\, S\, k$$

$$+ (1 + \mathrm{signum}(\%3))\, \pi^2 \sin(\phi)\, S\, g\, k + \%5\, \pi^2 \cos(\phi)\, S\, g\, k - (1 + \%4)\, \pi^2 \sin(\phi)\, S\, k$$

$$- \mathrm{signum}(\%3)\, \pi \cos(\phi)\, k^2 \lambda + \mathrm{signum}(\%3)\, \pi^2 \cos(\phi)\, S\, k + \frac{1}{2} (1 + \%4)\, \pi \sin(\phi)\, k^2 \lambda$$

$$+ (1 + \mathrm{signum}(\%1))\, \pi^2 \sin(\phi)\, S\, k \Big) \Big/ \pi^2 + \frac{1}{8} \sin(\phi)\, (-2\, k^2 \lambda^2 \ln(\pi\, S\, g - k\, \lambda - \pi\, S)$$

$$+ 4\, k^2 \lambda^2 (\ln(k) + I\, \pi) - 2\, k^2 \lambda^2 \ln(-\pi\, S\, g - k\, \lambda + \pi\, S) - 2\, \pi\, S\, g\, k\, \lambda \ln(\pi\, S\, g - k\, \lambda + \pi\, S)$$

$$- 2\, \pi\, S\, k\, \lambda \ln(\pi\, S\, g - k\, \lambda + \pi\, S) - 2\, \pi\, S\, g\, k\, \lambda \ln(-\pi\, S\, g - k\, \lambda + \pi\, S)$$

$$+ 2\, \pi\, S\, k\, \lambda \ln(-\pi\, S\, g - k\, \lambda + \pi\, S) + 2\, \pi\, S\, k\, \lambda \ln(-\pi\, S\, g - k\, \lambda - \pi\, S) - 2\, \pi\, S\, k\, \lambda \ln(-2\, \pi\, S - k\, \lambda)$$

$$- 2\, \pi\, S\, g\, k\, \lambda \ln(-2\, \pi\, S\, g - k\, \lambda) + 2\, \pi\, S\, g\, k\, \lambda \ln(\pi\, S\, g - k\, \lambda - \pi\, S)$$

$$- 2\, \pi\, S\, k\, \lambda \ln(\pi\, S\, g - k\, \lambda - \pi\, S) + 2\, \pi\, S\, g\, k\, \lambda \ln(-\pi\, S\, g - k\, \lambda - \pi\, S) + 2\, \pi\, S\, k\, \lambda \ln(2\, \pi\, S - k\, \lambda)$$

$$- k^2 \lambda^2 \ln(-2\, \pi\, S - k\, \lambda) + 4\, k^2 \lambda^2 \ln(\lambda) - k^2 \lambda^2 \ln(2\, \pi\, S\, g - k\, \lambda) - k^2 \lambda^2 \ln(-2\, \pi\, S\, g - k\, \lambda)$$

$$+ 2\, k^2 \lambda^2 \ln(-\pi\, S\, g - k\, \lambda - \pi\, S) + 2\, k^2 \lambda^2 \ln(\pi\, S\, g - k\, \lambda + \pi\, S) - k^2 \lambda^2 \ln(2\, \pi\, S - k\, \lambda)$$

$$+ 2\, \pi\, S\, g\, k\, \lambda \ln(2\, \pi\, S\, g - k\, \lambda) + 4\, \pi^2 S^2 + 4\, \pi^2 S^2 g^2 - 8\, \pi^2 S^2 g) \Big/ \pi^2 - \frac{1}{8} \sin(\phi)\, ($$

$$-2\, k^2 \lambda^2 \ln(\%3) + 4\, k^2 \lambda^2 \ln(k) + 2\, k^2 \lambda^2 \ln(\pi\, S\, g + k\, \lambda + \pi\, S) - 2\, k^2 \lambda^2 \ln(\%1)$$

$$+ 2\, \pi\, S\, k\, \lambda \ln(-2\, \pi\, S + k\, \lambda) - k^2 \lambda^2 \ln(-2\, \pi\, S + k\, \lambda) - 2\, \pi\, S\, g\, k\, \lambda \ln(\%3) + 2\, \pi\, S\, k\, \lambda \ln(\%3)$$

$$- 2\, \pi\, S\, k\, \lambda \ln(\%2) + 2\, k^2 \lambda^2 \ln(\%2) - 2\, \pi\, S\, g\, k\, \lambda \ln(\%2) - k^2 \lambda^2 \ln(-2\, \pi\, S\, g + k\, \lambda)$$

$$+ 2\, \pi\, S\, g\, k\, \lambda \ln(-2\, \pi\, S\, g + k\, \lambda) + 2\, \pi\, S\, g\, k\, \lambda \ln(\%1) - k^2 \lambda^2 \ln(2\, \pi\, S\, g + k\, \lambda)$$

$$- 2\, \pi\, S\, k\, \lambda \ln(\%1) - 2\, \pi\, S\, g\, k\, \lambda \ln(2\, \pi\, S\, g + k\, \lambda) + 2\, \pi\, S\, g\, k\, \lambda \ln(\pi\, S\, g + k\, \lambda + \pi\, S)$$

$$+ 2\, \pi\, S\, k\, \lambda \ln(\pi\, S\, g + k\, \lambda + \pi\, S) - 2\, \pi\, S\, k\, \lambda \ln(2\, \pi\, S + k\, \lambda) - k^2 \lambda^2 \ln(2\, \pi\, S + k\, \lambda)$$

$$+ 4\, k^2 \lambda^2 \ln(\lambda) + 4\, \pi^2 S^2 + 4\, \pi^2 S^2 g^2 - 8\, \pi^2 S^2 g) \Big/ \pi^2$$

$$\%1 := -\pi\, S\, g + k\, \lambda + \pi\, S$$

$$\%2 := -\pi\, S\, g + k\, \lambda - \pi\, S$$

$$\%3 := \pi\, S\, g + k\, \lambda - \pi\, S$$

$$\%4 := \mathrm{signum}(-2\, \pi\, S + k\, \lambda)$$

$$\%5 := \mathrm{signum}(-2\, \pi\, S\, g + k\, \lambda)$$

```
cost(");
```

283 *additions* + 921 *multiplications* + 4 *divisions* + 157 *functions*

1.3. Simplification of the Result.

Well, that's a mess. Let's see how much we can clean it up.

```
Iv := collect(Iv,[sin(phi),cos(phi),signum,ln],factor);
```

$$
\begin{aligned}
Iv := \Bigg(& -\frac{1}{8}\frac{\lambda\, k\,(2\,\pi\, S-k\,\lambda)\,\mathrm{signum}(-2\,\pi\, S+k\,\lambda)}{\pi} \\
& -\frac{1}{8}\frac{\lambda\, k\,(2\,\pi\, S\, g-k\,\lambda)\,\mathrm{signum}(-2\,\pi\, S\, g+k\,\lambda)}{\pi}+\frac{1}{4}\frac{\lambda\, k\,\%3\,\mathrm{signum}(-\pi\, S\, g+k\,\lambda-\pi\, S)}{\pi} \\
& -\frac{1}{4}\frac{\lambda\, k\,\%2\,\mathrm{signum}(-\pi\, S\, g+k\,\lambda+\pi\, S)}{\pi}+\frac{1}{4}\frac{\lambda\, k\,\%1\,\mathrm{signum}(\%1)}{\pi}+\frac{1}{4}\frac{\lambda\, k\,\%1\,\ln(\%1)}{\pi^2} \\
& -\frac{1}{4}\frac{k\,\lambda\,(\pi\, S\, g+k\,\lambda+\pi\, S)\,\ln(\pi\, S\, g+k\,\lambda+\pi\, S)}{\pi^2}-\frac{1}{4}\frac{\lambda\, k\,\%2\,\ln(-\pi\, S\, g+k\,\lambda+\pi\, S)}{\pi^2} \\
& +\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S+k\,\lambda)\,\ln(2\,\pi\, S+k\,\lambda)}{\pi^2}+\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S\, g+k\,\lambda)\,\ln(2\,\pi\, S\, g+k\,\lambda)}{\pi^2} \\
& -\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S-k\,\lambda)\,\ln(-2\,\pi\, S+k\,\lambda)}{\pi^2}+\frac{1}{4}\frac{k\,\lambda\,\%3\,\ln(-\pi\, S\, g+k\,\lambda-\pi\, S)}{\pi^2} \\
& -\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S\, g-k\,\lambda)\,\ln(-2\,\pi\, S\, g+k\,\lambda)}{\pi^2}+\frac{1}{4}\frac{\lambda\, k\,\%2\,\ln(\%2)}{\pi^2} \\
& +\frac{1}{4}\frac{k\,\lambda\,(\pi\, S\, g+k\,\lambda+\pi\, S)\,\ln(-\pi\, S\, g-k\,\lambda-\pi\, S)}{\pi^2}-\frac{1}{4}\frac{k\,\lambda\,\%3\,\ln(\%3)}{\pi^2} \\
& +\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S\, g-k\,\lambda)\,\ln(2\,\pi\, S\, g-k\,\lambda)}{\pi^2}-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S\, g+k\,\lambda)\,\ln(-2\,\pi\, S\, g-k\,\lambda)}{\pi^2} \\
& -\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S+k\,\lambda)\,\ln(-2\,\pi\, S-k\,\lambda)}{\pi^2}-\frac{1}{4}\frac{\lambda\, k\,\%1\,\ln(-\pi\, S\, g-k\,\lambda+\pi\, S)}{\pi^2} \\
& +\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\, S-k\,\lambda)\,\ln(2\,\pi\, S-k\,\lambda)}{\pi^2}\Bigg)\sin(\phi)+\Bigg(\frac{1}{8}\frac{\lambda\, k\,(2\,\pi\, S-k\,\lambda)\,\mathrm{signum}(-2\,\pi\, S+k\,\lambda)}{\pi} \\
& +\frac{1}{8}\frac{\lambda\, k\,(2\,\pi\, S\, g-k\,\lambda)\,\mathrm{signum}(-2\,\pi\, S\, g+k\,\lambda)}{\pi}-\frac{1}{4}\frac{\lambda\, k\,\%3\,\mathrm{signum}(-\pi\, S\, g+k\,\lambda-\pi\, S)}{\pi} \\
& +\frac{1}{4}\frac{\lambda\, k\,\%2\,\mathrm{signum}(-\pi\, S\, g+k\,\lambda+\pi\, S)}{\pi}-\frac{1}{4}\frac{\lambda\, k\,\%1\,\mathrm{signum}(\%1)}{\pi}+\frac{1}{2}\frac{\lambda^2\, k^2}{\pi}\Bigg)\cos(\phi)
\end{aligned}
$$

$$\%1 := \pi\, S\, g+k\,\lambda-\pi\, S$$

$$\%2 := \pi\, S\, g-k\,\lambda-\pi\, S$$

$$\%3 := \pi S g - k \lambda + \pi S$$

```
cost(");
```

$$106\ \textit{additions} + 322\ \textit{multiplications} + 27\ \textit{divisions} + 28\ \textit{functions}$$

```
Iv := convert(Iv,abs);
```

$$Iv := \left(-\frac{1}{8} \frac{\lambda k (2 \pi S - k \lambda) \left| -2 \pi S + k \lambda \right|}{\pi (-2 \pi S + k \lambda)} - \frac{1}{8} \frac{\lambda k (2 \pi S g - k \lambda) \left| -2 \pi S g + k \lambda \right|}{\pi (-2 \pi S g + k \lambda)} \right.$$

$$+ \frac{1}{4} \frac{\lambda k \%5 \left| \%4 \right|}{\pi \%4} - \frac{1}{4} \frac{\lambda k \%3 \left| \%2 \right|}{\pi \%2} + \frac{1}{4} \frac{\lambda k \left| \%1 \right|}{\pi} + \frac{1}{4} \frac{\lambda k \%1 \ln(\%1)}{\pi^2}$$

$$- \frac{1}{4} \frac{k \lambda (\pi S g + k \lambda + \pi S) \ln(\pi S g + k \lambda + \pi S)}{\pi^2} - \frac{1}{4} \frac{\lambda k \%3 \ln(\%2)}{\pi^2}$$

$$+ \frac{1}{8} \frac{k \lambda (2 \pi S + k \lambda) \ln(2 \pi S + k \lambda)}{\pi^2} + \frac{1}{8} \frac{k \lambda (2 \pi S g + k \lambda) \ln(2 \pi S g + k \lambda)}{\pi^2}$$

$$- \frac{1}{8} \frac{k \lambda (2 \pi S - k \lambda) \ln(-2 \pi S + k \lambda)}{\pi^2} + \frac{1}{4} \frac{k \lambda \%5 \ln(\%4)}{\pi^2}$$

$$- \frac{1}{8} \frac{k \lambda (2 \pi S g - k \lambda) \ln(-2 \pi S g + k \lambda)}{\pi^2} + \frac{1}{4} \frac{\lambda k \%3 \ln(\%3)}{\pi^2}$$

$$+ \frac{1}{4} \frac{k \lambda (\pi S g + k \lambda + \pi S) \ln(-\pi S g - k \lambda - \pi S)}{\pi^2} - \frac{1}{4} \frac{k \lambda \%5 \ln(\%5)}{\pi^2}$$

$$+ \frac{1}{8} \frac{k \lambda (2 \pi S g - k \lambda) \ln(2 \pi S g - k \lambda)}{\pi^2} - \frac{1}{8} \frac{k \lambda (2 \pi S g + k \lambda) \ln(-2 \pi S g - k \lambda)}{\pi^2}$$

$$- \frac{1}{8} \frac{k \lambda (2 \pi S + k \lambda) \ln(-2 \pi S - k \lambda)}{\pi^2} - \frac{1}{4} \frac{\lambda k \%1 \ln(-\pi S g - k \lambda + \pi S)}{\pi^2}$$

$$\left. + \frac{1}{8} \frac{k \lambda (2 \pi S - k \lambda) \ln(2 \pi S - k \lambda)}{\pi^2} \right) \sin(\phi) + \left(\frac{1}{8} \frac{\lambda k (2 \pi S - k \lambda) \left| -2 \pi S + k \lambda \right|}{\pi (-2 \pi S + k \lambda)} \right.$$

$$+ \frac{1}{8} \frac{\lambda k (2 \pi S g - k \lambda) \left| -2 \pi S g + k \lambda \right|}{\pi (-2 \pi S g + k \lambda)} - \frac{1}{4} \frac{\lambda k \%5 \left| \%4 \right|}{\pi \%4} + \frac{1}{4} \frac{\lambda k \%3 \left| \%2 \right|}{\pi \%2} - \frac{1}{4} \frac{\lambda k \left| \%1 \right|}{\pi}$$

$$\left. + \frac{1}{2} \frac{\lambda^2 k^2}{\pi} \right) \cos(\phi)$$

$$\%1 := \pi S g + k \lambda - \pi S$$

$$\%2 := -\pi S g + k \lambda + \pi S$$

$$\%3 := \pi S g - k \lambda - \pi S$$

$$\%4 := -\pi S g + k \lambda - \pi S$$

$$\%5 := \pi S g - k \lambda + \pi S$$

Make the substitution $f = \frac{k\,\lambda}{\pi\,S}$. We'll do it in stages, via an intermediary, $Q = \frac{k\,\lambda}{\pi}$. First, factor out Q globally.

```
collect( algsubs(k*lambda/Pi=Q,Iv,exact),
         [sin(phi),cos(phi),Q,abs,ln] );
```

$$\left(-\frac{1}{8}\frac{|-2\,\pi\,S+k\,\lambda|\,(2\,\pi\,S-k\,\lambda)}{-2\,\pi\,S+k\,\lambda}-\frac{1}{8}\frac{|-2\,\pi\,S\,g+k\,\lambda|\,(2\,\pi\,S\,g-k\,\lambda)}{-2\,\pi\,S\,g+k\,\lambda}+\frac{1}{4}\frac{|\%4|\,\%5}{\%4}-\frac{1}{4}\frac{|\%2|\,\%3}{\%2}\right.$$
$$+\frac{1}{4}\frac{\ln(\%1)\,\%1}{\pi}-\frac{1}{4}\frac{\ln(\pi\,S\,g+k\,\lambda+\pi\,S)\,(\pi\,S\,g+k\,\lambda+\pi\,S)}{\pi}-\frac{1}{4}\frac{\ln(\%2)\,\%3}{\pi}$$
$$+\frac{1}{8}\frac{\ln(2\,\pi\,S+k\,\lambda)\,(2\,\pi\,S+k\,\lambda)}{\pi}+\frac{1}{8}\frac{\ln(2\,\pi\,S\,g+k\,\lambda)\,(2\,\pi\,S\,g+k\,\lambda)}{\pi}$$
$$-\frac{1}{8}\frac{\ln(-2\,\pi\,S+k\,\lambda)\,(2\,\pi\,S-k\,\lambda)}{\pi}+\frac{1}{4}\frac{\ln(\%4)\,\%5}{\pi}-\frac{1}{8}\frac{\ln(-2\,\pi\,S\,g+k\,\lambda)\,(2\,\pi\,S\,g-k\,\lambda)}{\pi}$$
$$+\frac{1}{4}\frac{\ln(\%3)\,\%3}{\pi}+\frac{1}{4}\frac{\ln(-\pi\,S\,g-k\,\lambda-\pi\,S)\,(\pi\,S\,g+k\,\lambda+\pi\,S)}{\pi}-\frac{1}{4}\frac{\ln(\%5)\,\%5}{\pi}$$
$$+\frac{1}{8}\frac{\ln(2\,\pi\,S\,g-k\,\lambda)\,(2\,\pi\,S\,g-k\,\lambda)}{\pi}-\frac{1}{8}\frac{\ln(-2\,\pi\,S\,g-k\,\lambda)\,(2\,\pi\,S\,g+k\,\lambda)}{\pi}$$
$$-\frac{1}{8}\frac{\ln(-2\,\pi\,S-k\,\lambda)\,(2\,\pi\,S+k\,\lambda)}{\pi}-\frac{1}{4}\frac{\ln(-\pi\,S\,g-k\,\lambda+\pi\,S)\,\%1}{\pi}$$
$$\left.+\frac{1}{8}\frac{\ln(2\,\pi\,S-k\,\lambda)\,(2\,\pi\,S-k\,\lambda)}{\pi}+\frac{1}{4}|\%1|\right)Q\sin(\phi)+\left(\frac{1}{8}\frac{|-2\,\pi\,S+k\,\lambda|\,(2\,\pi\,S-k\,\lambda)}{-2\,\pi\,S+k\,\lambda}\right.$$
$$\left.+\frac{1}{8}\frac{|-2\,\pi\,S\,g+k\,\lambda|\,(2\,\pi\,S\,g-k\,\lambda)}{-2\,\pi\,S\,g+k\,\lambda}-\frac{1}{4}\frac{|\%4|\,\%5}{\%4}+\frac{1}{4}\frac{|\%2|\,\%3}{\%2}-\frac{1}{4}|\%1|+\frac{1}{2}k\,\lambda\right)Q\cos(\phi)$$

$$\%1 := \pi\,S\,g+k\,\lambda-\pi\,S$$
$$\%2 := -\pi\,S\,g+k\,\lambda+\pi\,S$$
$$\%3 := \pi\,S\,g-k\,\lambda-\pi\,S$$
$$\%4 := -\pi\,S\,g+k\,\lambda-\pi\,S$$
$$\%5 := \pi\,S\,g-k\,\lambda+\pi\,S$$

Next, create the "inside" Q terms.

```
Iv := subs( k*lambda=Pi*Q, " );
```

$$Iv := \left(-\frac{1}{8}\frac{|-2\,\pi\,S+\pi\,Q|\,(2\,\pi\,S-\pi\,Q)}{-2\,\pi\,S+\pi\,Q}-\frac{1}{8}\frac{|-2\,\pi\,S\,g+\pi\,Q|\,(2\,\pi\,S\,g-\pi\,Q)}{-2\,\pi\,S\,g+\pi\,Q}\right.$$
$$+\frac{1}{4}\frac{|-\pi\,S\,g+\pi\,Q-\pi\,S|\,(\pi\,S\,g-\pi\,Q+\pi\,S)}{-\pi\,S\,g+\pi\,Q-\pi\,S}-\frac{1}{4}\frac{|-\pi\,S\,g+\pi\,Q+\pi\,S|\,(\pi\,S\,g-\pi\,Q-\pi\,S)}{-\pi\,S\,g+\pi\,Q+\pi\,S}$$
$$+\frac{1}{4}\frac{\ln(\pi\,S\,g+\pi\,Q-\pi\,S)\,(\pi\,S\,g+\pi\,Q-\pi\,S)}{\pi}-\frac{1}{4}\frac{\ln(\pi\,S\,g+\pi\,Q+\pi\,S)\,(\pi\,S\,g+\pi\,Q+\pi\,S)}{\pi}$$
$$-\frac{1}{4}\frac{\ln(-\pi\,S\,g+\pi\,Q+\pi\,S)\,(\pi\,S\,g-\pi\,Q-\pi\,S)}{\pi}+\frac{1}{8}\frac{\ln(2\,\pi\,S+\pi\,Q)\,(2\,\pi\,S+\pi\,Q)}{\pi}$$

$$+\frac{1}{8}\frac{\ln(2\pi S g+\pi Q)(2\pi S g+\pi Q)}{\pi}-\frac{1}{8}\frac{\ln(-2\pi S+\pi Q)(2\pi S-\pi Q)}{\pi}$$
$$+\frac{1}{4}\frac{\ln(-\pi S g+\pi Q-\pi S)(\pi S g-\pi Q+\pi S)}{\pi}-\frac{1}{8}\frac{\ln(-2\pi S g+\pi Q)(2\pi S g-\pi Q)}{\pi}$$
$$+\frac{1}{4}\frac{\ln(\pi S g-\pi Q-\pi S)(\pi S g-\pi Q-\pi S)}{\pi}+\frac{1}{4}\frac{\ln(-\pi S g-\pi Q-\pi S)(\pi S g+\pi Q+\pi S)}{\pi}$$
$$-\frac{1}{4}\frac{\ln(\pi S g-\pi Q+\pi S)(\pi S g-\pi Q+\pi S)}{\pi}+\frac{1}{8}\frac{\ln(2\pi S g-\pi Q)(2\pi S g-\pi Q)}{\pi}$$
$$-\frac{1}{8}\frac{\ln(-2\pi S g-\pi Q)(2\pi S g+\pi Q)}{\pi}-\frac{1}{8}\frac{\ln(-2\pi S-\pi Q)(2\pi S+\pi Q)}{\pi}$$
$$-\frac{1}{4}\frac{\ln(-\pi S g-\pi Q+\pi S)(\pi S g+\pi Q-\pi S)}{\pi}+\frac{1}{8}\frac{\ln(2\pi S-\pi Q)(2\pi S-\pi Q)}{\pi}$$
$$+\frac{1}{4}\left|\pi S g+\pi Q-\pi S\right|\Bigg)Q\sin(\phi)+\Bigg(\frac{1}{8}\frac{\left|-2\pi S+\pi Q\right|(2\pi S-\pi Q)}{-2\pi S+\pi Q}$$
$$+\frac{1}{8}\frac{\left|-2\pi S g+\pi Q\right|(2\pi S g-\pi Q)}{-2\pi S g+\pi Q}-\frac{1}{4}\frac{\left|-\pi S g+\pi Q-\pi S\right|(\pi S g-\pi Q+\pi S)}{-\pi S g+\pi Q-\pi S}$$
$$+\frac{1}{4}\frac{\left|-\pi S g+\pi Q+\pi S\right|(\pi S g-\pi Q-\pi S)}{-\pi S g+\pi Q+\pi S}-\frac{1}{4}\left|\pi S g+\pi Q-\pi S\right|+\frac{1}{2}\pi Q\Bigg)Q\cos(\phi)$$

Now we need to set $Q=f S$. Write a procedure to take care of all the *abs* and *ln* terms in place, without messing up the form of the expression..

```
Iv_factor := proc( expr )
  local p, loc, newexpr;
  newexpr := expr;
  if type(newexpr,function) then
    if op(0,newexpr)=`abs` then
      newexpr := factor(subs(Q=f*S,newexpr));
    elif op(0,newexpr)=`ln` then
      p := factor( subs( Q=f*S, op(1,newexpr) ) );
      newexpr := expand( ln(p) );
    fi;
  else
    if nops(newexpr) > 1 then
      for p in newexpr do
        loc := location( newexpr, p );
        if nops(loc) > 0 then
          p := procname(p);
          newexpr := subsop( loc=p, newexpr );
        fi;
      od;
    fi;
  fi;
  RETURN(newexpr);
end:
```

```
Iv_factor(Iv);
```

$$\Bigg(-\frac{1}{8}\frac{\pi S\left|-2+f\right|(2\pi S-\pi Q)}{-2\pi S+\pi Q}-\frac{1}{8}\frac{\pi S\left|-2 g+f\right|(2\pi S g-\pi Q)}{-2\pi S g+\pi Q}$$

$$+\frac{1}{4}\frac{\pi S\left|-g+f-1\right|(\pi S g-\pi Q+\pi S)}{-\pi S g+\pi Q-\pi S}-\frac{1}{4}\frac{\pi S\left|-g+f+1\right|(\pi S g-\pi Q-\pi S)}{-\pi S g+\pi Q+\pi S}$$

$$+\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(g+f-1))(\pi S g+\pi Q-\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(g+f+1))(\pi S g+\pi Q+\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(-g+f+1))(\pi S g-\pi Q-\pi S)}{\pi}$$

$$+\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(2+f))(2\pi S+\pi Q)}{\pi}+\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(2g+f))(2\pi S g+\pi Q)}{\pi}$$

$$-\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(-2+f))(2\pi S-\pi Q)}{\pi}$$

$$+\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(-g+f-1))(\pi S g-\pi Q+\pi S)}{\pi}$$

$$-\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(-2g+f))(2\pi S g-\pi Q)}{\pi}$$

$$+\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(-1+g-f))(\pi S g-\pi Q-\pi S)}{\pi}$$

$$+\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(-g-f-1))(\pi S g+\pi Q+\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(g-f+1))(\pi S g-\pi Q+\pi S)}{\pi}$$

$$+\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(2g-f))(2\pi S g-\pi Q)}{\pi}$$

$$-\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(-2g-f))(2\pi S g+\pi Q)}{\pi}$$

$$-\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(-2-f))(2\pi S+\pi Q)}{\pi}$$

$$-\frac{1}{4}\frac{(\ln(\pi)+\ln(S)+\ln(-g-f+1))(\pi S g+\pi Q-\pi S)}{\pi}$$

$$+\frac{1}{8}\frac{(\ln(\pi)+\ln(S)+\ln(2-f))(2\pi S-\pi Q)}{\pi}+\frac{1}{4}\pi S\left|g+f-1\right|\Bigg)Q\sin(\phi)+\Bigg($$

$$\frac{1}{8}\frac{\pi S\left|-2+f\right|(2\pi S-\pi Q)}{-2\pi S+\pi Q}+\frac{1}{8}\frac{\pi S\left|-2g+f\right|(2\pi S g-\pi Q)}{-2\pi S g+\pi Q}$$

$$-\frac{1}{4}\frac{\pi S\left|-g+f-1\right|(\pi S g-\pi Q+\pi S)}{-\pi S g+\pi Q-\pi S}+\frac{1}{4}\frac{\pi S\left|-g+f+1\right|(\pi S g-\pi Q-\pi S)}{-\pi S g+\pi Q+\pi S}$$

$$-\frac{1}{4}\pi S\left|g+f-1\right|+\frac{1}{2}\pi Q\Bigg)Q\cos(\phi)$$

Finally, finish off the remaining Q substitutions. Many of the *ln* terms conveniently go away

now.

```
Iv := collect( subs(Q=f*S,"), [sin(phi),cos(phi),S], factor );
```

$$
\begin{aligned}
Iv := -\frac{1}{8}\big(&2\ln(-1+g-f)-2\ln(-g-f-1)+2\ln(g-f+1)+2\ln(g+f-1)-\ln(-2g+f)f \\
&-\ln(2g+f)f-\ln(-2+f)f-2\ln(-g+f-1)g+2\ln(-g+f-1)f-2\ln(-g+f+1)f \\
&-2\ln(g+f-1)f+2\ln(g+f+1)g-2\ln(g+f-1)g-2\ln(-1+g-f)g+2\ln(-1+g-f)f \\
&+2\ln(-g-f+1)f+\ln(2-f)f+2\ln(-g-f+1)g+\ln(-2-f)f+2\ln(-2g-f)g \\
&+\ln(-2g-f)f+\ln(2g-f)f+2\ln(g-f+1)g-2\ln(g-f+1)f-2\ln(2g-f)g \\
&+2\ln(-2g+f)g-\pi|-2+f|-\pi|-2g+f|+2\pi|-g+f-1|-2\pi|-g+f+1| \\
&-2\ln(-g-f-1)g-2\ln(-g-f-1)f-\ln(2+f)f-2\ln(2g+f)g+2\ln(g+f+1)f \\
&+2\ln(-g+f+1)g+2\ln(g+f+1)+2\ln(-2-f)-2\ln(-g-f+1)-2\ln(-g+f+1) \\
&-2\ln(2+f)-2\ln(2-f)+2\ln(-2+f)-2\ln(-g+f-1)-2\pi|g+f-1|\big)fS^2\sin(\phi) \\
&+\frac{1}{8}\pi\big(-|-2+f|-|-2g+f|+2|-g+f-1|-2|-g+f+1|-2|g+f-1|+4f\big)fS^2\cos(\phi)
\end{aligned}
$$

```
cost(");
```

$$130\ \textit{additions} + 93\ \textit{multiplications} + 52\ \textit{functions}$$

Much better. Notice there are both $\sin\phi$ and $\cos\phi$ terms.

```
Iv_sin := coeff(Iv,sin(phi),1)*sin(phi);
```

$$
\begin{aligned}
Iv_sin := -\frac{1}{8}\big(&2\ln(-1+g-f)-2\ln(-g-f-1)+2\ln(g-f+1)+2\ln(g+f-1) \\
&-\ln(-2g+f)f-\ln(2g+f)f-\ln(-2+f)f-2\ln(-g+f-1)g+2\ln(-g+f-1)f \\
&-2\ln(-g+f+1)f-2\ln(g+f-1)f+2\ln(g+f+1)g-2\ln(g+f-1)g-2\ln(-1+g-f)g \\
&+2\ln(-1+g-f)f+2\ln(-g-f+1)f+\ln(2-f)f+2\ln(-g-f+1)g+\ln(-2-f)f \\
&+2\ln(-2g-f)g+\ln(-2g-f)f+\ln(2g-f)f+2\ln(g-f+1)g-2\ln(g-f+1)f \\
&-2\ln(2g-f)g+2\ln(-2g+f)g-\pi|-2+f|-\pi|-2g+f|+2\pi|-g+f-1|-2\pi|-g+f+1| \\
&-2\ln(-g-f-1)g-2\ln(-g-f-1)f-\ln(2+f)f-2\ln(2g+f)g+2\ln(g+f+1)f \\
&+2\ln(-g+f+1)g+2\ln(g+f+1)+2\ln(-2-f)-2\ln(-g-f+1)-2\ln(-g+f+1) \\
&-2\ln(2+f)-2\ln(2-f)+2\ln(-2+f)-2\ln(-g+f-1)-2\pi|g+f-1|\big)fS^2\sin(\phi)
\end{aligned}
$$

```
Iv_cos := coeff(Iv,cos(phi),1)*cos(phi);
```

$$
Iv_cos := \frac{1}{8}\pi\big(-|-2+f|-|-2g+f|+2|-g+f-1|-2|-g+f+1|-2|g+f-1|+4f\big)fS^2\cos(\phi)
$$

1.4. Characterization of the *sin* and *cos* Terms.

Let's make the coefficients of $\frac{\pi S^2\sin(\phi)}{8}$ and $\frac{\pi S^2\cos(\phi)}{8}$ into functions of (g,f).

```
G[s] := fn( coeff( algsubs( S^2*sin(phi)=Q, Iv_sin ), Q, 1 )*8/Pi,
            g, f );
```

$$G_s := (g,f) \to -(2\ln(-g-f+1)f + 2\ln(-1+g-f)f - 2\ln(-1+g-f)g - 2\ln(g+f-1)g$$
$$+ 2\ln(g+f+1)g - 2\ln(g+f-1)f - 2\ln(-g+f+1)f + 2\ln(-g+f-1)f - 2\ln(-g+f-1)g$$
$$- \ln(-2+f)f - \ln(2g+f)f - \ln(-2g+f)f - 2\ln(-g-f-1)g - 2\pi|-g+f+1|$$
$$+ 2\pi|-g+f-1| - \pi|-2g+f| - \pi|-2+f| + 2\ln(-2g+f)g - 2\ln(2g-f)g - 2\ln(g-f+1)f$$
$$+ 2\ln(g-f+1)g + \ln(2g-f)f + \ln(-2g-f)f + 2\ln(-2g-f)g + \ln(-2-f)f$$
$$+ 2\ln(-g-f+1)g + \ln(2-f)f - \ln(2+f)f - 2\ln(-g-f-1)f - 2\pi|g+f-1|$$
$$+ 2\ln(-g+f+1)g + 2\ln(g+f+1)f - 2\ln(2g+f)g + 2\ln(g+f-1) + 2\ln(g-f+1)$$
$$- 2\ln(-g-f-1) + 2\ln(-1+g-f) + 2\ln(-2-f) - 2\ln(2-f) - 2\ln(-g-f+1)$$
$$+ 2\ln(g+f+1) - 2\ln(-g+f+1) - 2\ln(-g+f-1) + 2\ln(-2+f) - 2\ln(2+f))f/\pi$$

```
G[c] := fn( coeff( algsubs( S^2*cos(phi)=Q, Iv_cos ), Q, 1 )*8/Pi,
            g, f );
```

$$G_c := (g,f) \to (-|-2+f| - |-2g+f| + 2|-g+f-1| - 2|-g+f+1| - 2|g+f-1| + 4f)f$$

Notice that the collection of *ln* terms in the *sin* term make for a very difficult time of keeping the *sin* term purely real. In fact, setting $g=0$ we have $G_s(0,f)$

$$-(-2\pi|1+f| - 2\ln(f)f + 2\ln(-f)f - \pi|f| - \ln(-2+f)f + \ln(2-f)f + \ln(-2-f)f - \pi|-2+f|$$
$$- \ln(2+f)f + 2\ln(-2-f) - 2\ln(2+f) - 2\ln(2-f) + 2\ln(-2+f))f/\pi$$

```
collect(",[abs,f],factor);
```

$$2f|1+f| + f|f| + f|-2+f|$$
$$- \frac{(2\ln(-f) - \ln(-2+f) - 2\ln(f) + \ln(-2-f) - \ln(2+f) + \ln(2-f))f^2}{\pi}$$
$$- 2\frac{(\ln(-2-f) - \ln(2+f) - \ln(2-f) + \ln(-2+f))f}{\pi}$$

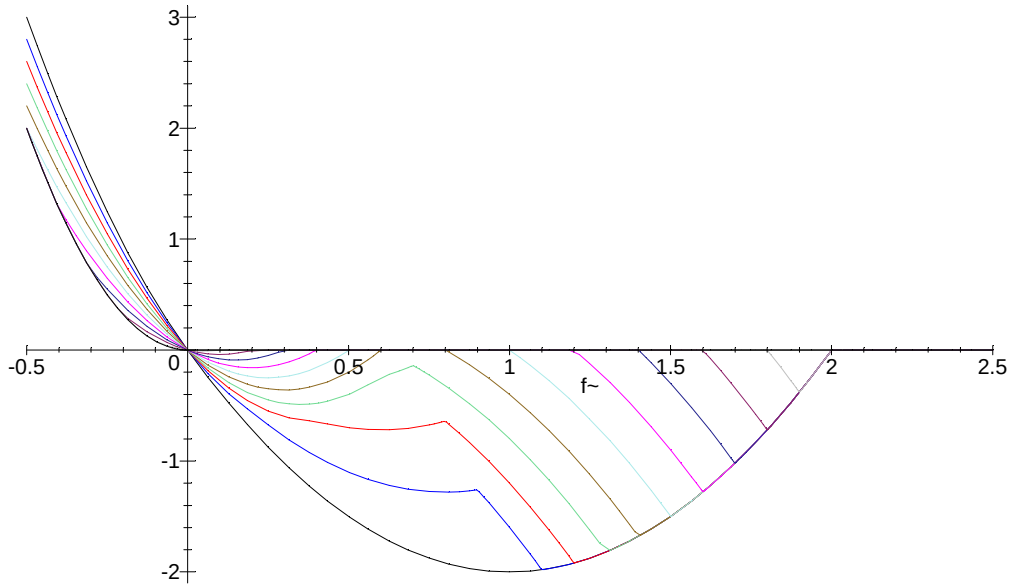
where it becomes quite clear we're in trouble. There is no real value of f for which this expression does not have an imaginary component. On the other hand, the *cos* term is comparatively well-behaved: $G_c(g,f)$

$$(-|-2+f| - |-2g+f| + 2|-g+f-1| - 2|-g+f+1| - 2|g+f-1| + 4f)f$$

Let's take a look at the *cos* term for various values of g.

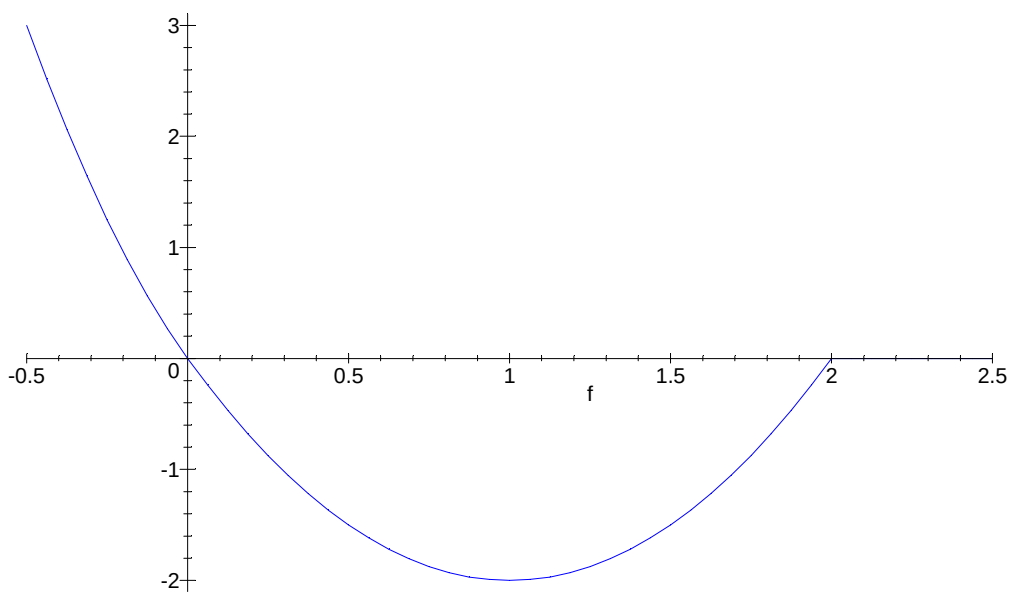
```
cosplot := proc( gvals::list, frange::range )
  local p, k;
  p := [];
  for k from 1 to nops(gvals) do
    p := [ op(p), plot( G[c](gvals[k],f), f=frange,
                        color=mycolors[(k-1 mod 10)+1] ) ];
  od;
  plots[display](p);
end:
```

```
cosplot( [seq(0.1*i,i=0..10)], -0.5..2.5 );
```



An animation of this is also instructive.

```
plots[animate](G[c](g,f),f=-0.5..2.5,g=0..1,color=blue);
```



Hmm, kind of disturbing that it is asymmetric.

2. Integration Check.

Since Mathcad 6.0 was unable to do the I_v integral and therefore provide an independent check, we should differentiate the integral and make sure we recover the integrand. First, do the indefinite integral.

```
Iv_indef := int( integrand, p );
```

$$
\begin{aligned}
Iv_indef := \Bigg(& -\frac{1}{4} \\
& \left(-\frac{\cos\left(\frac{(2\pi S + k\lambda)p}{\lambda} + \phi\right)\lambda}{(2\pi S + k\lambda)p} + \mathrm{Si}\left(-\frac{(2\pi S + k\lambda)p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{(2\pi S + k\lambda)p}{\lambda}\right)\sin(\phi)\right) \\
& (2\pi S + k\lambda)S/\lambda + \frac{1}{4} \\
& \left(-\frac{\cos\left(\frac{(-2\pi S + k\lambda)p}{\lambda} + \phi\right)\lambda}{(-2\pi S + k\lambda)p} + \mathrm{Si}\left(-\frac{(-2\pi S + k\lambda)p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{(-2\pi S + k\lambda)p}{\lambda}\right)\sin(\phi)\right) \\
& (-2\pi S + k\lambda)S/\lambda \\
& -\frac{1}{4}\frac{\left(-\frac{\cos\left(\frac{\%5\,p}{\lambda} + \phi\right)\lambda}{\%5\,p} + \mathrm{Si}\left(-\frac{\%5\,p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{\%5\,p}{\lambda}\right)\sin(\phi)\right)\%5\,S(1+g)}{\lambda} \\
& -\frac{1}{4}\frac{\left(-\frac{\cos\left(\frac{\%6\,p}{\lambda} + \phi\right)\lambda}{\%6\,p} + \mathrm{Si}\left(-\frac{\%6\,p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{\%6\,p}{\lambda}\right)\sin(\phi)\right)\%6\,S(g-1)}{\lambda} \\
& +\frac{1}{4}\frac{\left(-\frac{\cos\left(\frac{\%4\,p}{\lambda} + \phi\right)\lambda}{\%4\,p} + \mathrm{Si}\left(-\frac{\%4\,p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{\%4\,p}{\lambda}\right)\sin(\phi)\right)\%4\,S(1+g)}{\lambda} \\
& +\frac{1}{4}\frac{\left(-\frac{\cos\left(\frac{\%3\,p}{\lambda} + \phi\right)\lambda}{\%3\,p} + \mathrm{Si}\left(-\frac{\%3\,p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{\%3\,p}{\lambda}\right)\sin(\phi)\right)\%3\,S(g-1)}{\lambda} \\
& +\frac{1}{4}\frac{\left(-\frac{\cos(\%1 + \phi)\lambda}{(-2\pi S g + k\lambda)p} + \mathrm{Si}(-\%1)\cos(\phi) - \mathrm{Ci}(\%1)\sin(\phi)\right)(-2\pi S g + k\lambda)S g}{\lambda}
\end{aligned}
$$

$$-\frac{1}{4}\frac{\left(-\frac{\cos(\%2+\phi)\,\lambda}{(2\,\pi\,S\,g+k\,\lambda)\,p}+\mathrm{Si}(-\%2)\cos(\phi)-\mathrm{Ci}(\%2)\sin(\phi)\right)(2\,\pi\,S\,g+k\,\lambda)\,S\,g}{\lambda}$$

$$-\frac{\left(-\frac{1}{2}\frac{\sin(k\,p+\phi)}{k^2\,p^2}-\frac{1}{2}\frac{\cos(k\,p+\phi)}{k\,p}+\frac{1}{2}\mathrm{Si}(-k\,p)\cos(\phi)-\frac{1}{2}\mathrm{Ci}(k\,p)\sin(\phi)\right)k^2\,\lambda}{\pi}+\frac{1}{4}\Bigg($$

$$-\frac{1}{2}\frac{\sin\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)\lambda^2}{(2\,\pi\,S+k\,\lambda)^2\,p^2}-\frac{1}{2}\frac{\cos\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)\lambda}{(2\,\pi\,S+k\,\lambda)\,p}+\frac{1}{2}\mathrm{Si}\left(-\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)$$

$$-\frac{1}{2}\mathrm{Ci}\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)\Bigg)(2\,\pi\,S+k\,\lambda)^2/(\lambda\,\pi)+\frac{1}{4}\Bigg(-\frac{1}{2}\frac{\sin\left(\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)\lambda^2}{(-2\,\pi\,S+k\,\lambda)^2\,p^2}$$

$$-\frac{1}{2}\frac{\cos\left(\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)\lambda}{(-2\,\pi\,S+k\,\lambda)\,p}+\frac{1}{2}\mathrm{Si}\left(-\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)$$

$$-\frac{1}{2}\mathrm{Ci}\left(\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)\Bigg)(-2\,\pi\,S+k\,\lambda)^2/(\lambda\,\pi)+\frac{1}{2}$$

$$\frac{\left(-\frac{1}{2}\frac{\sin\left(\frac{\%6\,p}{\lambda}+\phi\right)\lambda^2}{\%6^2\,p^2}-\frac{1}{2}\frac{\cos\left(\frac{\%6\,p}{\lambda}+\phi\right)\lambda}{\%6\,p}+\frac{1}{2}\mathrm{Si}\left(-\frac{\%6\,p}{\lambda}\right)\cos(\phi)-\frac{1}{2}\mathrm{Ci}\left(\frac{\%6\,p}{\lambda}\right)\sin(\phi)\right)\%6^2}{\lambda\,\pi}$$

$$-\frac{1}{2}$$

$$\frac{\left(-\frac{1}{2}\frac{\sin\left(\frac{\%5\,p}{\lambda}+\phi\right)\lambda^2}{\%5^2\,p^2}-\frac{1}{2}\frac{\cos\left(\frac{\%5\,p}{\lambda}+\phi\right)\lambda}{\%5\,p}+\frac{1}{2}\mathrm{Si}\left(-\frac{\%5\,p}{\lambda}\right)\cos(\phi)-\frac{1}{2}\mathrm{Ci}\left(\frac{\%5\,p}{\lambda}\right)\sin(\phi)\right)\%5^2}{\lambda\,\pi}$$

$$-\frac{1}{2}$$

$$\frac{\left(-\frac{1}{2}\frac{\sin\left(\frac{\%4\,p}{\lambda}+\phi\right)\lambda^2}{\%4^2\,p^2}-\frac{1}{2}\frac{\cos\left(\frac{\%4\,p}{\lambda}+\phi\right)\lambda}{\%4\,p}+\frac{1}{2}\mathrm{Si}\left(-\frac{\%4\,p}{\lambda}\right)\cos(\phi)-\frac{1}{2}\mathrm{Ci}\left(\frac{\%4\,p}{\lambda}\right)\sin(\phi)\right)\%4^2}{\lambda\,\pi}$$

$$+\frac{1}{2}$$

$$\frac{\left(-\frac{1}{2}\frac{\sin\left(\frac{\%3\,p}{\lambda}+\phi\right)\lambda^2}{\%3^2\,p^2}-\frac{1}{2}\frac{\cos\left(\frac{\%3\,p}{\lambda}+\phi\right)\lambda}{\%3\,p}+\frac{1}{2}\,\mathrm{Si}\left(-\frac{\%3\,p}{\lambda}\right)\cos(\phi)-\frac{1}{2}\,\mathrm{Ci}\left(\frac{\%3\,p}{\lambda}\right)\sin(\phi)\right)\%3^2}{\lambda\,\pi}$$

$$+\frac{1}{4}\left(-\frac{1}{2}\frac{\sin(\%2+\phi)\,\lambda^2}{(2\,\pi\,S\,g+k\,\lambda)^2\,p^2}-\frac{1}{2}\frac{\cos(\%2+\phi)\,\lambda}{(2\,\pi\,S\,g+k\,\lambda)\,p}+\frac{1}{2}\,\mathrm{Si}(-\%2)\cos(\phi)-\frac{1}{2}\,\mathrm{Ci}(\%2)\sin(\phi)\right)$$

$$(2\,\pi\,S\,g+k\,\lambda)^2/(\lambda\,\pi)+\frac{1}{4}$$

$$\left(-\frac{1}{2}\frac{\sin(\%1+\phi)\,\lambda^2}{(-2\,\pi\,S\,g+k\,\lambda)^2\,p^2}-\frac{1}{2}\frac{\cos(\%1+\phi)\,\lambda}{(-2\,\pi\,S\,g+k\,\lambda)\,p}+\frac{1}{2}\,\mathrm{Si}(-\%1)\cos(\phi)-\frac{1}{2}\,\mathrm{Ci}(\%1)\sin(\phi)\right)$$

$$(-2\,\pi\,S\,g+k\,\lambda)^2/(\lambda\,\pi)\Bigg)\lambda/\pi$$

$$\%1 := \frac{(-2\,\pi\,S\,g+k\,\lambda)\,p}{\lambda}$$

$$\%2 := \frac{(2\,\pi\,S\,g+k\,\lambda)\,p}{\lambda}$$

$$\%3 := -\pi\,S\,g+k\,\lambda+\pi\,S$$

$$\%4 := \pi\,S\,g+k\,\lambda+\pi\,S$$

$$\%5 := -\pi\,S\,g+k\,\lambda-\pi\,S$$

$$\%6 := \pi\,S\,g+k\,\lambda-\pi\,S$$

```
cost(");
```

233 *additions* + 598 *multiplications* + 134 *divisions* + 94 *functions*

Now differentiate this wrt p.

```
diff( Iv_indef, p );
```

$$\left(-\frac{1}{4}\left(\frac{\sin(\%8)}{p}+\frac{\cos(\%8)\,\lambda}{(2\,\pi\,S+k\,\lambda)\,p^2}-\frac{\sin\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)}{p}\right)\right.$$

$$(2\,\pi\,S+k\,\lambda)\,S/\lambda+\frac{1}{4}$$

$$\left(\frac{\sin(\%7)}{p}+\frac{\cos(\%7)\,\lambda}{(-2\,\pi\,S+k\,\lambda)\,p^2}-\frac{\sin\left(\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{(-2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)}{p}\right)$$

$$(-2\,\pi\,S+k\,\lambda)\,S/\lambda$$

$$-\frac{1}{4}\,\frac{\left(\frac{\sin\left(\frac{\%5\,p}{\lambda}+\phi\right)}{p}+\frac{\cos\left(\frac{\%5\,p}{\lambda}+\phi\right)\lambda}{\%5\,p^2}-\frac{\sin\left(\frac{\%5\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{\%5\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%5\,S\,(1+g)}{\lambda}$$

$$-\frac{1}{4}\,\frac{\left(\frac{\sin\left(\frac{\%6\,p}{\lambda}+\phi\right)}{p}+\frac{\cos\left(\frac{\%6\,p}{\lambda}+\phi\right)\lambda}{\%6\,p^2}-\frac{\sin\left(\frac{\%6\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{\%6\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%6\,S\,(g-1)}{\lambda}$$

$$+\frac{1}{4}\,\frac{\left(\frac{\sin\left(\frac{\%4\,p}{\lambda}+\phi\right)}{p}+\frac{\cos\left(\frac{\%4\,p}{\lambda}+\phi\right)\lambda}{\%4\,p^2}-\frac{\sin\left(\frac{\%4\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{\%4\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%4\,S\,(1+g)}{\lambda}$$

$$+\frac{1}{4}\,\frac{\left(\frac{\sin\left(\frac{\%3\,p}{\lambda}+\phi\right)}{p}+\frac{\cos\left(\frac{\%3\,p}{\lambda}+\phi\right)\lambda}{\%3\,p^2}-\frac{\sin\left(\frac{\%3\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{\%3\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%3\,S\,(g-1)}{\lambda}$$

$$+\frac{1}{4}$$

$$\frac{\left(\frac{\sin(\%1+\phi)}{p}+\frac{\cos(\%1+\phi)\,\lambda}{(-2\,\pi\,S\,g+k\,\lambda)\,p^2}-\frac{\sin(\%1)\cos(\phi)}{p}-\frac{\cos(\%1)\sin(\phi)}{p}\right)(-2\,\pi\,S\,g+k\,\lambda)\,S\,g}{\lambda}$$

$$-\frac{1}{4}\,\frac{\left(\frac{\sin(\%2+\phi)}{p}+\frac{\cos(\%2+\phi)\,\lambda}{(2\,\pi\,S\,g+k\,\lambda)\,p^2}-\frac{\sin(\%2)\cos(\phi)}{p}-\frac{\cos(\%2)\sin(\phi)}{p}\right)(2\,\pi\,S\,g+k\,\lambda)\,S\,g}{\lambda}$$

$$-\frac{\left(\frac{\sin(k\,p+\phi)}{k^2\,p^3}+\frac{1}{2}\,\frac{\sin(k\,p+\phi)}{p}-\frac{1}{2}\,\frac{\sin(k\,p)\cos(\phi)}{p}-\frac{1}{2}\,\frac{\cos(k\,p)\sin(\phi)}{p}\right)k^2\,\lambda}{\pi}+\frac{1}{4}$$

$$\left(\frac{\sin(\%8)\,\lambda^2}{(2\,\pi\,S+k\,\lambda)^2\,p^3}+\frac{1}{2}\,\frac{\sin(\%8)}{p}-\frac{1}{2}\,\frac{\sin\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\,\frac{\cos\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)}{p}\right)$$

$$(2\pi S+k\lambda)^2/(\lambda\pi)+\frac{1}{4}\left(\frac{\sin(\%7)\lambda^2}{(-2\pi S+k\lambda)^2 p^3}+\frac{1}{2}\frac{\sin(\%7)}{p}-\frac{1}{2}\frac{\sin\left(\frac{(-2\pi S+k\lambda)p}{\lambda}\right)\cos(\phi)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\cos\left(\frac{(-2\pi S+k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right)(-2\pi S+k\lambda)^2/(\lambda\pi)$$

$$+\frac{1}{2}\frac{\left(\frac{\sin\left(\frac{\%6\,p}{\lambda}+\phi\right)\lambda^2}{\%6^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{\%6\,p}{\lambda}+\phi\right)}{p}-\frac{1}{2}\frac{\sin\left(\frac{\%6\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{\%6\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%6^2}{\lambda\pi}$$

$$-\frac{1}{2}\frac{\left(\frac{\sin\left(\frac{\%5\,p}{\lambda}+\phi\right)\lambda^2}{\%5^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{\%5\,p}{\lambda}+\phi\right)}{p}-\frac{1}{2}\frac{\sin\left(\frac{\%5\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{\%5\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%5^2}{\lambda\pi}$$

$$-\frac{1}{2}\frac{\left(\frac{\sin\left(\frac{\%4\,p}{\lambda}+\phi\right)\lambda^2}{\%4^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{\%4\,p}{\lambda}+\phi\right)}{p}-\frac{1}{2}\frac{\sin\left(\frac{\%4\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{\%4\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%4^2}{\lambda\pi}$$

$$+\frac{1}{2}\frac{\left(\frac{\sin\left(\frac{\%3\,p}{\lambda}+\phi\right)\lambda^2}{\%3^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{\%3\,p}{\lambda}+\phi\right)}{p}-\frac{1}{2}\frac{\sin\left(\frac{\%3\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{\%3\,p}{\lambda}\right)\sin(\phi)}{p}\right)\%3^2}{\lambda\pi}+$$

$$\frac{1}{4}\frac{\left(\frac{\sin(\%2+\phi)\lambda^2}{(2\pi S g+k\lambda)^2 p^3}+\frac{1}{2}\frac{\sin(\%2+\phi)}{p}-\frac{1}{2}\frac{\sin(\%2)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos(\%2)\sin(\phi)}{p}\right)(2\pi S g+k\lambda)^2}{\lambda\pi}$$

$$+\frac{1}{4}\left(\frac{\sin(\%1+\phi)\lambda^2}{(-2\pi S g+k\lambda)^2 p^3}+\frac{1}{2}\frac{\sin(\%1+\phi)}{p}-\frac{1}{2}\frac{\sin(\%1)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos(\%1)\sin(\phi)}{p}\right)$$

$$\left.(-2\pi S g+k\lambda)^2/(\lambda\pi)\right)\lambda/\pi$$

$$\%1 := \frac{(-2\pi S g+k\lambda)p}{\lambda}$$

$$\%2 := \frac{(2\,\pi\,S\,g + k\,\lambda)\,p}{\lambda}$$

$$\%3 := -\pi\,S\,g + k\,\lambda + \pi\,S$$

$$\%4 := \pi\,S\,g + k\,\lambda + \pi\,S$$

$$\%5 := -\pi\,S\,g + k\,\lambda - \pi\,S$$

$$\%6 := \pi\,S\,g + k\,\lambda - \pi\,S$$

$$\%7 := \frac{(-2\,\pi\,S + k\,\lambda)\,p}{\lambda} + \phi$$

$$\%8 := \frac{(2\,\pi\,S + k\,\lambda)\,p}{\lambda} + \phi$$

```
cost(");
```

$$249\ \mathit{additions} + 606\ \mathit{multiplications} + 175\ \mathit{divisions} + 102\ \mathit{functions}$$

This had better be equal to the original integrand.

```
expand( simplify(""-integrand) );
```

$$0$$

Whew. However, this means only that Maple is able to successfully differentiate the result of an integration. It does not necessarily mean the integration was correct, though it increases that probability.

3. Another Approach to Evaluating the Integral.

3.1. Simplification of the Indefinite Integral.

Start with the indefinite integration result and then insert the limits for p. First, let's simplify the form of the indefinite integral result. Recall that the indefinite integral from the last section is

```
Iv_indef := collect(Iv_indef,[sin(phi),cos(phi),p,lambda,g],factor);
```

$$\mathit{Iv_indef} := \left(-\frac{1}{8}k^2\left(2\,\mathrm{Ci}(-\%3) + 2\,\mathrm{Ci}(\%2) - 2\,\mathrm{Ci}(-\%4) + \mathrm{Ci}(\%5) + \mathrm{Ci}\left(-\frac{(2\,\pi\,S - k\,\lambda)\,p}{\lambda}\right)\right.\right.$$
$$\left. + \mathrm{Ci}\left(\frac{(2\,\pi\,S + k\,\lambda)\,p}{\lambda}\right) - 2\,\mathrm{Ci}(\%1) - 4\,\mathrm{Ci}(k\,p) + \mathrm{Ci}(-\%6)\right)\lambda^2 \Big/ \pi^2 + \Bigg($$
$$\frac{1}{4}\,\frac{k\,S\,(-\mathrm{Ci}(\%5) + \mathrm{Ci}(-\%3) - \mathrm{Ci}(-\%4) + \mathrm{Ci}(\%1) - \mathrm{Ci}(\%2) + \mathrm{Ci}(-\%6))\,g}{\pi} + \frac{1}{4}$$
$$\frac{k\,S\left(\mathrm{Ci}\left(-\frac{(2\,\pi\,S - k\,\lambda)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(2\,\pi\,S + k\,\lambda)\,p}{\lambda}\right) + \mathrm{Ci}(\%2) - \mathrm{Ci}(-\%3) - \mathrm{Ci}(-\%4) + \mathrm{Ci}(\%1)\right)}{\pi}$$

$$\left.\left.\right)\lambda\right)\sin(\phi)+\left(\frac{1}{8}k^2\left(-\mathrm{Si}\left(\frac{(2\pi S+k\lambda)p}{\lambda}\right)+4\,\mathrm{Si}(kp)+2\,\mathrm{Si}(\%3)-2\,\mathrm{Si}(\%4)+2\,\mathrm{Si}(\%1)\right.\right.$$

$$\left.-2\,\mathrm{Si}(\%2)-\mathrm{Si}(\%5)+\mathrm{Si}\left(\frac{(2\pi S-k\lambda)p}{\lambda}\right)+\mathrm{Si}(\%6)\right)\lambda^2\Big/\pi^2+\left(\right.$$

$$\frac{1}{4}\frac{kS(-\mathrm{Si}(\%3)+\mathrm{Si}(\%4)-\mathrm{Si}(\%6)-\mathrm{Si}(\%2)-\mathrm{Si}(\%5)+\mathrm{Si}(\%1))g}{\pi}$$

$$\left.+\frac{1}{4}\frac{kS\left(-\mathrm{Si}\left(\frac{(2\pi S-k\lambda)p}{\lambda}\right)+\mathrm{Si}(\%4)+\mathrm{Si}(\%3)-\mathrm{Si}\left(\frac{(2\pi S+k\lambda)p}{\lambda}\right)+\mathrm{Si}(\%2)+\mathrm{Si}(\%1)\right)}{\pi}\right.$$

$$\left.\left.\right)\lambda\right)\cos(\phi)+\frac{1}{8}k\left(2\cos\left(\frac{p\pi Sg-pk\lambda+p\pi S-\phi\lambda}{\lambda}\right)+2\cos\left(\frac{p\pi Sg+pk\lambda+p\pi S+\phi\lambda}{\lambda}\right)\right.$$

$$-2\cos\left(\frac{p\pi Sg-pk\lambda-p\pi S-\phi\lambda}{\lambda}\right)-\cos\left(\frac{2p\pi S-pk\lambda-\phi\lambda}{\lambda}\right)$$

$$-2\cos\left(\frac{p\pi Sg+pk\lambda-p\pi S+\phi\lambda}{\lambda}\right)-\cos\left(\frac{2p\pi Sg+pk\lambda+\phi\lambda}{\lambda}\right)$$

$$\left.-\cos\left(\frac{2p\pi S+pk\lambda+\phi\lambda}{\lambda}\right)+4\cos(kp+\phi)-\cos\left(\frac{2p\pi Sg-pk\lambda-\phi\lambda}{\lambda}\right)\right)\lambda^2\Big/(\pi^2p)+\frac{1}{8}$$

$$\left(4\sin(kp+\phi)+\sin\left(\frac{2p\pi S-pk\lambda-\phi\lambda}{\lambda}\right)-2\sin\left(\frac{p\pi Sg+pk\lambda-p\pi S+\phi\lambda}{\lambda}\right)\right.$$

$$+2\sin\left(\frac{p\pi Sg-pk\lambda-p\pi S-\phi\lambda}{\lambda}\right)-2\sin\left(\frac{p\pi Sg-pk\lambda+p\pi S-\phi\lambda}{\lambda}\right)$$

$$+2\sin\left(\frac{p\pi Sg+pk\lambda+p\pi S+\phi\lambda}{\lambda}\right)-\sin\left(\frac{2p\pi S+pk\lambda+\phi\lambda}{\lambda}\right)$$

$$\left.+\sin\left(\frac{2p\pi Sg-pk\lambda-\phi\lambda}{\lambda}\right)-\sin\left(\frac{2p\pi Sg+pk\lambda+\phi\lambda}{\lambda}\right)\right)\lambda^2\Big/(\pi^2p^2)$$

$$\%1:=\frac{(\pi Sg+k\lambda+\pi S)p}{\lambda}$$

$$\%2:=\frac{(\pi Sg+k\lambda-\pi S)p}{\lambda}$$

$$\%3:=\frac{(\pi Sg-k\lambda-\pi S)p}{\lambda}$$

$$\%4:=\frac{(\pi Sg-k\lambda+\pi S)p}{\lambda}$$

$$\%5:=\frac{(2\pi Sg+k\lambda)p}{\lambda}$$

$$\%6 := \frac{(2\,\pi\,S\,g - k\,\lambda)\,p}{\lambda}$$

Once again, we make the substitution $f = \frac{k\,\lambda}{\pi\,S}$. Additionally, we make the substitution $Q = \frac{k\,p}{f}$. Q replaces p as our independent variable. Here is a procedure that does the substitutions in the function arguments (*sin*, *cos*, *Ci*, *Si*).

```
Iv_indef_factor := proc( expr )
  local q, loc, newexpr;
  newexpr := expr;
  if type(newexpr,function) then
    if member( op(0,newexpr), {`sin`,`cos`,`Ci`,`Si`} ) then
      q := op(1,newexpr);
      q := factor( subs( S=k*lambda/Pi/f, q ) );
      q := collect( algsubs( k*p='Q'*f, q ), Q, factor );
      newexpr := op(0,newexpr)(q);
    fi;
  else
    if nops(newexpr) > 1 then
      for q in newexpr do
        loc := location( newexpr, q );
        if nops(loc) > 0 then
          q := procname(q);
          newexpr := subsop( loc=q, newexpr );
        fi;
      od;
    fi;
  fi;
  RETURN(newexpr);
end:
```

```
Iv_indef_factor( Iv_indef );
```

$$\Big(-\frac{1}{8}k^2\,(2\,\mathrm{Ci}((-g+f+1)\,Q) + 2\,\mathrm{Ci}((g+f-1)\,Q) - 2\,\mathrm{Ci}((-g+f-1)\,Q) + \mathrm{Ci}((2\,g+f)\,Q)$$
$$+\,\mathrm{Ci}((-2+f)\,Q) + \mathrm{Ci}((2+f)\,Q) - 2\,\mathrm{Ci}((g+f+1)\,Q) - 4\,\mathrm{Ci}(Q\,f) + \mathrm{Ci}((-2\,g+f)\,Q))\,\lambda^2 \Big/$$
$$\pi^2 + \Big(\frac{1}{4}k\,S\,(-\mathrm{Ci}((2\,g+f)\,Q) + \mathrm{Ci}((-g+f+1)\,Q) - \mathrm{Ci}((-g+f-1)\,Q) + \mathrm{Ci}((g+f+1)\,Q)$$
$$-\,\mathrm{Ci}((g+f-1)\,Q) + \mathrm{Ci}((-2\,g+f)\,Q))\,g/\pi + \frac{1}{4}k\,S\,(\mathrm{Ci}((-2+f)\,Q) - \mathrm{Ci}((2+f)\,Q)$$
$$+\,\mathrm{Ci}((g+f-1)\,Q) - \mathrm{Ci}((-g+f+1)\,Q) - \mathrm{Ci}((-g+f-1)\,Q) + \mathrm{Ci}((g+f+1)\,Q))/\pi\Big)\lambda\Big)$$
$$\sin(\phi) + \Big(\frac{1}{8}k^2\,(-\mathrm{Si}((2+f)\,Q) + 4\,\mathrm{Si}(Q\,f) + 2\,\mathrm{Si}((-1+g-f)\,Q) - 2\,\mathrm{Si}((g-f+1)\,Q)$$
$$+\,2\,\mathrm{Si}((g+f+1)\,Q) - 2\,\mathrm{Si}((g+f-1)\,Q) - \mathrm{Si}((2\,g+f)\,Q) + \mathrm{Si}((2-f)\,Q) + \mathrm{Si}((2\,g-f)\,Q))$$
$$\lambda^2 \Big/ \pi^2 + \Big(\frac{1}{4}k\,S\,(-\mathrm{Si}((-1+g-f)\,Q) + \mathrm{Si}((g-f+1)\,Q) - \mathrm{Si}((2\,g-f)\,Q)$$
$$-\,\mathrm{Si}((g+f-1)\,Q) - \mathrm{Si}((2\,g+f)\,Q) + \mathrm{Si}((g+f+1)\,Q))\,g/\pi + \frac{1}{4}k\,S\,(-\mathrm{Si}((2-f)\,Q)$$

$$+\mathrm{Si}((g-f+1)Q)+\mathrm{Si}((-1+g-f)Q)-\mathrm{Si}((2+f)Q)+\mathrm{Si}((g+f-1)Q)$$
$$+\mathrm{Si}((g+f+1)Q))/\pi\Big)\lambda\Big)\cos(\phi)+\frac{1}{8}k(2\cos(-(g-f+1)Q+\phi)+2\cos((g+f+1)Q+\phi)$$
$$-2\cos(-(-1+g-f)Q+\phi)-\cos(-(2-f)Q+\phi)-2\cos((g+f-1)Q+\phi)$$
$$-\cos((2g+f)Q+\phi)-\cos((2+f)Q+\phi)+4\cos(Qf+\phi)-\cos(-(2g-f)Q+\phi))\lambda^2\Big/($$
$$\pi^2 p)+\frac{1}{8}(4\sin(Qf+\phi)-\sin(-(2-f)Q+\phi)-2\sin((g+f-1)Q+\phi)$$
$$-2\sin(-(-1+g-f)Q+\phi)+2\sin(-(g-f+1)Q+\phi)+2\sin((g+f+1)Q+\phi)$$
$$-\sin((2+f)Q+\phi)-\sin(-(2g-f)Q+\phi)-\sin((2g+f)Q+\phi))\lambda^2\Big/(\pi^2 p^2)$$

Now make the substitutions on the "outside" variables.

```
subs( p=Q*f/k, S=k*lambda/Pi/f, " );
```

$$\Big(-\frac{1}{8}k^2(2\,\mathrm{Ci}((-g+f+1)Q)+2\,\mathrm{Ci}((g+f-1)Q)-2\,\mathrm{Ci}((-g+f-1)Q)+\mathrm{Ci}((2g+f)Q)$$
$$+\mathrm{Ci}((-2+f)Q)+\mathrm{Ci}((2+f)Q)-2\,\mathrm{Ci}((g+f+1)Q)-4\,\mathrm{Ci}(Qf)+\mathrm{Ci}((-2g+f)Q))\lambda^2\Big/$$
$$\pi^2+\Big(\frac{1}{4}k^2\lambda(-\mathrm{Ci}((2g+f)Q)+\mathrm{Ci}((-g+f+1)Q)-\mathrm{Ci}((-g+f-1)Q)+\mathrm{Ci}((g+f+1)Q)$$
$$-\mathrm{Ci}((g+f-1)Q)+\mathrm{Ci}((-2g+f)Q))g\Big/(\pi^2 f)+\frac{1}{4}k^2\lambda(\mathrm{Ci}((-2+f)Q)-\mathrm{Ci}((2+f)Q)$$
$$+\mathrm{Ci}((g+f-1)Q)-\mathrm{Ci}((-g+f+1)Q)-\mathrm{Ci}((-g+f-1)Q)+\mathrm{Ci}((g+f+1)Q))\Big/(\pi^2 f)\Big)$$
$$\lambda\Big)\sin(\phi)+\Big(\frac{1}{8}k^2(-\mathrm{Si}((2+f)Q)+4\,\mathrm{Si}(Qf)+2\,\mathrm{Si}((-1+g-f)Q)-2\,\mathrm{Si}((g-f+1)Q)$$
$$+2\,\mathrm{Si}((g+f+1)Q)-2\,\mathrm{Si}((g+f-1)Q)-\mathrm{Si}((2g+f)Q)+\mathrm{Si}((2-f)Q)+\mathrm{Si}((2g-f)Q))$$
$$\lambda^2\Big/\pi^2+\Big(\frac{1}{4}k^2\lambda(-\mathrm{Si}((-1+g-f)Q)+\mathrm{Si}((g-f+1)Q)-\mathrm{Si}((2g-f)Q)$$
$$-\mathrm{Si}((g+f-1)Q)-\mathrm{Si}((2g+f)Q)+\mathrm{Si}((g+f+1)Q))g\Big/(\pi^2 f)+\frac{1}{4}k^2\lambda(-\mathrm{Si}((2-f)Q)$$
$$+\mathrm{Si}((g-f+1)Q)+\mathrm{Si}((-1+g-f)Q)-\mathrm{Si}((2+f)Q)+\mathrm{Si}((g+f-1)Q)$$
$$+\mathrm{Si}((g+f+1)Q))\Big/(\pi^2 f)\Big)\lambda\Big)\cos(\phi)+\frac{1}{8}k^2(2\cos(-(g-f+1)Q+\phi)$$
$$+2\cos((g+f+1)Q+\phi)-2\cos(-(-1+g-f)Q+\phi)-\cos(-(2-f)Q+\phi)$$
$$-2\cos((g+f-1)Q+\phi)-\cos((2g+f)Q+\phi)-\cos((2+f)Q+\phi)+4\cos(Qf+\phi)$$
$$-\cos(-(2g-f)Q+\phi))\lambda^2\Big/(\pi^2 Qf)+\frac{1}{8}(4\sin(Qf+\phi)-\sin(-(2-f)Q+\phi)$$
$$-2\sin((g+f-1)Q+\phi)-2\sin(-(-1+g-f)Q+\phi)+2\sin(-(g-f+1)Q+\phi)$$
$$+2\sin((g+f+1)Q+\phi)-\sin((2+f)Q+\phi)-\sin(-(2g-f)Q+\phi)-\sin((2g+f)Q+\phi))$$

$$\lambda^2 k^2 \Big/ (\pi^2 Q^2 f^2)$$

```
Iv_indef := collect( ", [sin(phi),cos(phi),Q,Pi,f,k,lambda] );
```

$$
\begin{aligned}
Iv_indef := \Bigg(\Bigg(&-\frac{1}{4}\mathrm{Ci}((-g+f+1)\,Q)-\frac{1}{4}\mathrm{Ci}((g+f-1)\,Q)+\frac{1}{4}\mathrm{Ci}((-g+f-1)\,Q)\\
&-\frac{1}{8}\mathrm{Ci}((2\,g+f)\,Q)-\frac{1}{8}\mathrm{Ci}((-2+f)\,Q)-\frac{1}{8}\mathrm{Ci}((2+f)\,Q)+\frac{1}{4}\mathrm{Ci}((g+f+1)\,Q)+\frac{1}{2}\mathrm{Ci}(Q\,f)\\
&-\frac{1}{8}\mathrm{Ci}((-2\,g+f)\,Q)\Bigg)\lambda^2 k^2+\Bigg(\frac{1}{4}(-\mathrm{Ci}((2\,g+f)\,Q)+\mathrm{Ci}((-g+f+1)\,Q)-\mathrm{Ci}((-g+f-1)\,Q)\\
&+\mathrm{Ci}((g+f+1)\,Q)-\mathrm{Ci}((g+f-1)\,Q)+\mathrm{Ci}((-2\,g+f)\,Q))\,g+\frac{1}{4}\mathrm{Ci}((-2+f)\,Q)\\
&-\frac{1}{4}\mathrm{Ci}((2+f)\,Q)+\frac{1}{4}\mathrm{Ci}((g+f-1)\,Q)-\frac{1}{4}\mathrm{Ci}((-g+f+1)\,Q)-\frac{1}{4}\mathrm{Ci}((-g+f-1)\,Q)\\
&+\frac{1}{4}\mathrm{Ci}((g+f+1)\,Q)\Bigg)\lambda^2 k^2/f\Bigg)\sin(\phi)\Big/\pi^2+\Bigg(\Bigg(-\frac{1}{8}\mathrm{Si}((2+f)\,Q)+\frac{1}{2}\mathrm{Si}(Q\,f)\\
&+\frac{1}{4}\mathrm{Si}((-1+g-f)\,Q)-\frac{1}{4}\mathrm{Si}((g-f+1)\,Q)+\frac{1}{4}\mathrm{Si}((g+f+1)\,Q)-\frac{1}{4}\mathrm{Si}((g+f-1)\,Q)\\
&-\frac{1}{8}\mathrm{Si}((2\,g+f)\,Q)+\frac{1}{8}\mathrm{Si}((2-f)\,Q)+\frac{1}{8}\mathrm{Si}((2\,g-f)\,Q)\Bigg)\lambda^2 k^2+\Bigg(\frac{1}{4}(-\mathrm{Si}((-1+g-f)\,Q)\\
&+\mathrm{Si}((g-f+1)\,Q)-\mathrm{Si}((2\,g-f)\,Q)-\mathrm{Si}((g+f-1)\,Q)-\mathrm{Si}((2\,g+f)\,Q)+\mathrm{Si}((g+f+1)\,Q))\\
&g-\frac{1}{4}\mathrm{Si}((2-f)\,Q)+\frac{1}{4}\mathrm{Si}((g-f+1)\,Q)+\frac{1}{4}\mathrm{Si}((-1+g-f)\,Q)-\frac{1}{4}\mathrm{Si}((2+f)\,Q)\\
&+\frac{1}{4}\mathrm{Si}((g+f-1)\,Q)+\frac{1}{4}\mathrm{Si}((g+f+1)\,Q)\Bigg)\lambda^2 k^2/f\Bigg)\cos(\phi)\Big/\pi^2+\Bigg(\\
&\frac{1}{4}\cos(-(g-f+1)\,Q+\phi)+\frac{1}{4}\cos((g+f+1)\,Q+\phi)-\frac{1}{4}\cos(-(-1+g-f)\,Q+\phi)\\
&-\frac{1}{8}\cos(-(2-f)\,Q+\phi)-\frac{1}{4}\cos((g+f-1)\,Q+\phi)-\frac{1}{8}\cos((2\,g+f)\,Q+\phi)\\
&-\frac{1}{8}\cos((2+f)\,Q+\phi)+\frac{1}{2}\cos(Q\,f+\phi)-\frac{1}{8}\cos(-(2\,g-f)\,Q+\phi)\Bigg)\lambda^2 k^2\Big/(f\,\pi^2\,Q)+\Bigg(\\
&\frac{1}{2}\sin(Q\,f+\phi)-\frac{1}{8}\sin(-(2-f)\,Q+\phi)-\frac{1}{4}\sin((g+f-1)\,Q+\phi)-\frac{1}{4}\sin(-(-1+g-f)\,Q+\phi)\\
&+\frac{1}{4}\sin(-(g-f+1)\,Q+\phi)+\frac{1}{4}\sin((g+f+1)\,Q+\phi)-\frac{1}{8}\sin((2+f)\,Q+\phi)\\
&-\frac{1}{8}\sin(-(2\,g-f)\,Q+\phi)-\frac{1}{8}\sin((2\,g+f)\,Q+\phi)\Bigg)\lambda^2 k^2\Big/(f^2\,\pi^2\,Q^2)
\end{aligned}
$$

3.2. Substitution of the Integration Limits into the Indefinite Integral Expression.

Now do the integration limits $Q=-\infty$ to $Q=\infty$.

```
L1 := collect( limit( Iv_indef, Q=-infinity ),
               [sin(phi),cos(phi),Pi,f] );
```

$$
\begin{aligned}
L1 := \Bigg(&\frac{1}{8}\lambda^2 k^2\Bigg(-\frac{1}{2}\,\mathrm{signum}(-2+f)-\mathrm{signum}(g-f+1)-\mathrm{signum}(-g-f-1)+2\,\mathrm{signum}(f)\\
&+\mathrm{signum}(-1+g-f)+\frac{1}{2}\,\mathrm{signum}(-2\,g-f)+\mathrm{signum}(-g-f+1)+\frac{1}{2}\,\mathrm{signum}(2\,g-f)\\
&-\frac{1}{2}\,\mathrm{signum}(2+f)\Bigg)+\frac{1}{8}\lambda^2 k^2\,(\mathrm{signum}(-1+g-f)-\mathrm{signum}(-g-f-1)\\
&+g\,(-1+\mathrm{signum}(-g-f+1))-\mathrm{signum}(2+f)+g\,(-1+\mathrm{signum}(-2\,g-f))+\mathrm{signum}(-2+f)\\
&-g\,(-1+\mathrm{signum}(-g-f-1))-g\,(-1+\mathrm{signum}(-1+g-f))+g\,(-1+\mathrm{signum}(g-f+1))\\
&-\mathrm{signum}(-g-f+1)-g\,(-1+\mathrm{signum}(2\,g-f))+\mathrm{signum}(g-f+1))/f\Bigg)\sin(\phi)/\pi+\Bigg(\frac{1}{8}\lambda^2 k^2\\
&\Bigg(-\mathrm{signum}(-g+f-1)+\frac{1}{2}\,\mathrm{signum}(-2+f)+\mathrm{signum}(-g+f+1)-\mathrm{signum}(g+f+1)\\
&+\frac{1}{2}\,\mathrm{signum}(2\,g+f)-2\,\mathrm{signum}(f)+\frac{1}{2}\,\mathrm{signum}(-2\,g+f)+\frac{1}{2}\,\mathrm{signum}(2+f)+\mathrm{signum}(g+f-1)\Bigg)\\
&+\frac{1}{8}\lambda^2 k^2\,(g\,\mathrm{signum}(2\,g+f)-g\,\mathrm{signum}(-g+f+1)-\mathrm{signum}(-2+f)+\mathrm{signum}(2+f)\\
&-g\,\mathrm{signum}(g+f+1)-\mathrm{signum}(g+f-1)-\mathrm{signum}(g+f+1)+\mathrm{signum}(-g+f-1)\\
&+g\,\mathrm{signum}(g+f-1)+\mathrm{signum}(-g+f+1)+g\,\mathrm{signum}(-g+f-1)-g\,\mathrm{signum}(-2\,g+f))/f\Bigg)\\
&\cos(\phi)/\pi
\end{aligned}
$$

```
L2 := collect( limit( Iv_indef, Q=infinity ),
               [sin(phi),cos(phi),Pi,f] );
```

$$
\begin{aligned}
L2 := \Bigg(&\frac{1}{8}\lambda^2 k^2\Bigg(-\mathrm{signum}(-g+f-1)+\frac{1}{2}\,\mathrm{signum}(-2+f)+\mathrm{signum}(-g+f+1)\\
&-\mathrm{signum}(g+f+1)+\frac{1}{2}\,\mathrm{signum}(2\,g+f)-2\,\mathrm{signum}(f)+\frac{1}{2}\,\mathrm{signum}(-2\,g+f)+\frac{1}{2}\,\mathrm{signum}(2+f)\\
&+\mathrm{signum}(g+f-1)\Bigg)+\frac{1}{8}\lambda^2 k^2\,(\mathrm{signum}(-g+f+1)+g\,(-1+\mathrm{signum}(2\,g+f))\\
&-\mathrm{signum}(-2+f)+g\,(-1+\mathrm{signum}(g+f-1))-g\,(-1+\mathrm{signum}(-g+f+1))\\
&-\mathrm{signum}(g+f-1)-g\,(-1+\mathrm{signum}(-2\,g+f))+\mathrm{signum}(-g+f-1)\\
&-g\,(-1+\mathrm{signum}(g+f+1))+g\,(-1+\mathrm{signum}(-g+f-1))+\mathrm{signum}(2+f)\\
&-\mathrm{signum}(g+f+1))/f\Bigg)\sin(\phi)/\pi+\Bigg(\frac{1}{8}\lambda^2 k^2\Bigg(\mathrm{signum}(g+f+1)-\frac{1}{2}\,\mathrm{signum}(-2+f)\\
&-\mathrm{signum}(-g+f+1)-\frac{1}{2}\,\mathrm{signum}(-2\,g+f)+\mathrm{signum}(-g+f-1)-\mathrm{signum}(g+f-1)
\end{aligned}
$$

$$-\frac{1}{2}\,\mathrm{signum}(2+f)-\frac{1}{2}\,\mathrm{signum}(2\,g+f)+2\,\mathrm{signum}(f)\Big)+\frac{1}{8}\,\lambda^2\,k^2\,(-g\,\mathrm{signum}(2\,g+f)$$
$$+\,\mathrm{signum}(-2+f)-\mathrm{signum}(2+f)-\mathrm{signum}(-g+f+1)+\mathrm{signum}(g+f+1)$$
$$+\,g\,\mathrm{signum}(g+f+1)+g\,\mathrm{signum}(-g+f+1)+\mathrm{signum}(g+f-1)-g\,\mathrm{signum}(-g+f-1)$$
$$-\,\mathrm{signum}(-g+f-1)-g\,\mathrm{signum}(g+f-1)+g\,\mathrm{signum}(-2\,g+f))\,/\,f\Big)\cos(\phi)\,/\,\pi$$

We obtain the result

```
Iv := collect( algsubs( k*lambda=Pi*f*S, simplify(L2-L1) ),
                [S,sin(phi),cos(phi),g] );
```

$$Iv := \Bigg(\Bigg(\frac{1}{16}\,(2\,\mathrm{signum}(g+f-1)-2\,\mathrm{signum}(g+f+1)-2\,\mathrm{signum}(-g+f+1)$$
$$+\,2\,\mathrm{signum}(2\,g-f)-2\,\mathrm{signum}(-2\,g+f)-2\,\mathrm{signum}(-2\,g-f)+2\,\mathrm{signum}(-g-f-1)$$
$$+\,2\,\mathrm{signum}(-1+g-f)-2\,\mathrm{signum}(g-f+1)-2\,\mathrm{signum}(-g-f+1)+2\,\mathrm{signum}(2\,g+f)$$
$$+\,2\,\mathrm{signum}(-g+f-1))\,\pi\,f\,g+\frac{1}{16}\,(-2\,\mathrm{signum}(g+f-1)-2\,\mathrm{signum}(g-f+1)$$
$$-\,2\,\mathrm{signum}(g+f+1)+2\,\mathrm{signum}(-g+f+1)+4\,\mathrm{signum}(2+f)-f\,\mathrm{signum}(2\,g-f)$$
$$-\,2\,f\,\mathrm{signum}(-g-f+1)-2\,f\,\mathrm{signum}(-1+g-f)+2\,f\,\mathrm{signum}(-g-f-1)+f\,\mathrm{signum}(2\,g+f)$$
$$+\,f\,\mathrm{signum}(-2\,g+f)+2\,f\,\mathrm{signum}(g-f+1)+2\,f\,\mathrm{signum}(-g+f+1)+2\,f\,\mathrm{signum}(-2+f)$$
$$-\,8\,f\,\mathrm{signum}(f)-f\,\mathrm{signum}(-2\,g-f)+2\,f\,\mathrm{signum}(2+f)-2\,f\,\mathrm{signum}(-g+f-1)$$
$$+\,2\,f\,\mathrm{signum}(g+f-1)-2\,f\,\mathrm{signum}(g+f+1)-4\,\mathrm{signum}(-2+f)+2\,\mathrm{signum}(-g+f-1)$$
$$-\,2\,\mathrm{signum}(-1+g-f)+2\,\mathrm{signum}(-g-f+1)+2\,\mathrm{signum}(-g-f-1))\,\pi\,f\Bigg)\sin(\phi)+\Bigg(\frac{1}{16}\,($$
$$-4\,\mathrm{signum}(g+f-1)-4\,\mathrm{signum}(2\,g+f)+4\,\mathrm{signum}(g+f+1)+4\,\mathrm{signum}(-g+f+1)$$
$$-\,4\,\mathrm{signum}(-g+f-1)+4\,\mathrm{signum}(-2\,g+f))\,\pi\,f\,g+\frac{1}{16}\,(-4\,\mathrm{signum}(2+f)$$
$$-\,4\,\mathrm{signum}(-g+f+1)-2\,f\,\mathrm{signum}(-2+f)+4\,\mathrm{signum}(g+f+1)+4\,\mathrm{signum}(g+f-1)$$
$$-\,2\,f\,\mathrm{signum}(2+f)-2\,f\,\mathrm{signum}(-2\,g+f)+8\,f\,\mathrm{signum}(f)-4\,f\,\mathrm{signum}(g+f-1)$$
$$+\,4\,f\,\mathrm{signum}(g+f+1)-4\,f\,\mathrm{signum}(-g+f+1)+4\,\mathrm{signum}(-2+f)-4\,\mathrm{signum}(-g+f-1)$$
$$+\,4\,f\,\mathrm{signum}(-g+f-1)-2\,f\,\mathrm{signum}(2\,g+f))\,\pi\,f\Bigg)\cos(\phi)\Bigg)S^2$$

Shit, this is not the same as in section 1. At least it does not contain those suspicious *ln* terms. The *sin* and *cos* parts are

```
Iv_sin := coeff( Iv, sin(phi) );
```

$$Iv_sin := \Bigg(\frac{1}{16}\,(2\,\mathrm{signum}(g+f-1)-2\,\mathrm{signum}(g+f+1)-2\,\mathrm{signum}(-g+f+1)$$
$$+\,2\,\mathrm{signum}(2\,g-f)-2\,\mathrm{signum}(-2\,g+f)-2\,\mathrm{signum}(-2\,g-f)+2\,\mathrm{signum}(-g-f-1)$$
$$+\,2\,\mathrm{signum}(-1+g-f)-2\,\mathrm{signum}(g-f+1)-2\,\mathrm{signum}(-g-f+1)+2\,\mathrm{signum}(2\,g+f)$$

$$+ 2\,\mathrm{signum}(-g+f-1))\,\pi f g + \frac{1}{16}(-2\,\mathrm{signum}(g+f-1) - 2\,\mathrm{signum}(g-f+1)$$
$$- 2\,\mathrm{signum}(g+f+1) + 2\,\mathrm{signum}(-g+f+1) + 4\,\mathrm{signum}(2+f) - f\,\mathrm{signum}(2g-f)$$
$$- 2f\,\mathrm{signum}(-g-f+1) - 2f\,\mathrm{signum}(-1+g-f) + 2f\,\mathrm{signum}(-g-f-1) + f\,\mathrm{signum}(2g+f)$$
$$+ f\,\mathrm{signum}(-2g+f) + 2f\,\mathrm{signum}(g-f+1) + 2f\,\mathrm{signum}(-g+f+1) + 2f\,\mathrm{signum}(-2+f)$$
$$- 8f\,\mathrm{signum}(f) - f\,\mathrm{signum}(-2g-f) + 2f\,\mathrm{signum}(2+f) - 2f\,\mathrm{signum}(-g+f-1)$$
$$+ 2f\,\mathrm{signum}(g+f-1) - 2f\,\mathrm{signum}(g+f+1) - 4\,\mathrm{signum}(-2+f) + 2\,\mathrm{signum}(-g+f-1)$$
$$- 2\,\mathrm{signum}(-1+g-f) + 2\,\mathrm{signum}(-g-f+1) + 2\,\mathrm{signum}(-g-f-1))\,\pi f\Big)S^2$$

```
Iv_cos := coeff( Iv, cos(phi) );
```

$$Iv_cos := \Big(\frac{1}{16}(-4\,\mathrm{signum}(g+f-1) - 4\,\mathrm{signum}(2g+f) + 4\,\mathrm{signum}(g+f+1)$$
$$+ 4\,\mathrm{signum}(-g+f+1) - 4\,\mathrm{signum}(-g+f-1) + 4\,\mathrm{signum}(-2g+f))\,\pi f g + \frac{1}{16}($$
$$-4\,\mathrm{signum}(2+f) - 4\,\mathrm{signum}(-g+f+1) - 2f\,\mathrm{signum}(-2+f) + 4\,\mathrm{signum}(g+f+1)$$
$$+ 4\,\mathrm{signum}(g+f-1) - 2f\,\mathrm{signum}(2+f) - 2f\,\mathrm{signum}(-2g+f) + 8f\,\mathrm{signum}(f)$$
$$- 4f\,\mathrm{signum}(g+f-1) + 4f\,\mathrm{signum}(g+f+1) - 4f\,\mathrm{signum}(-g+f+1) + 4\,\mathrm{signum}(-2+f)$$
$$- 4\,\mathrm{signum}(-g+f-1) + 4f\,\mathrm{signum}(-g+f-1) - 2f\,\mathrm{signum}(2g+f))\,\pi f\Big)S^2$$

3.3. Characterization of the *sin* and *cos* Terms.

As before, define the coefficients of $\frac{\pi S^2 \sin(\phi)}{8}$ and $\frac{\pi S^2 \cos(\phi)}{8}$ as functions of g and f.

```
G[s] := fn( collect(Iv_sin/Pi/S^2*8,[f,g]), g, f );
```

$$G_s := (g,f) \to \Big(\mathrm{signum}(2+f) + \mathrm{signum}(g+f-1) - \mathrm{signum}(-g+f-1) - \frac{1}{2}\mathrm{signum}(-2g-f)$$
$$- \mathrm{signum}(-1+g-f) - \mathrm{signum}(g+f+1) - 4\,\mathrm{signum}(f) + \frac{1}{2}\mathrm{signum}(-2g+f)$$
$$+ \mathrm{signum}(g-f+1) + \mathrm{signum}(-g-f-1) + \frac{1}{2}\mathrm{signum}(2g+f) + \mathrm{signum}(-g+f+1)$$
$$+ \mathrm{signum}(-2+f) - \frac{1}{2}\mathrm{signum}(2g-f) - \mathrm{signum}(-g-f+1)\Big)f^2 + ((\mathrm{signum}(g+f-1)$$
$$- \mathrm{signum}(g+f+1) - \mathrm{signum}(-g+f+1) + \mathrm{signum}(2g-f) - \mathrm{signum}(-2g+f)$$
$$- \mathrm{signum}(-2g-f) + \mathrm{signum}(-g-f-1) + \mathrm{signum}(-1+g-f) - \mathrm{signum}(g-f+1)$$
$$- \mathrm{signum}(-g-f+1) + \mathrm{signum}(2g+f) + \mathrm{signum}(-g+f-1))\,g + \mathrm{signum}(-g+f+1)$$
$$- \mathrm{signum}(g-f+1) + \mathrm{signum}(-g+f-1) - 2\,\mathrm{signum}(-2+f) - \mathrm{signum}(g+f-1)$$
$$- \mathrm{signum}(g+f+1) + 2\,\mathrm{signum}(2+f) + \mathrm{signum}(-g-f-1) + \mathrm{signum}(-g-f+1)$$
$$- \mathrm{signum}(-1+g-f))\,f$$

```
G[c] := fn( collect(Iv_cos/Pi/S^2*8,[f,g]), g, f );
```

$$G_c := (g,f) \to (-\mathrm{signum}(-2+f) - \mathrm{signum}(2+f) - \mathrm{signum}(-2\,g+f) + 2\,\mathrm{signum}(g+f+1) - 2\,\mathrm{signum}(-g+f+1) + 4\,\mathrm{signum}(f) - 2\,\mathrm{signum}(g+f-1) + 2\,\mathrm{signum}(-g+f-1) - \mathrm{signum}(2\,g+f))\,f^2 + ((-2\,\mathrm{signum}(g+f-1) - 2\,\mathrm{signum}(2\,g+f) + 2\,\mathrm{signum}(g+f+1) + 2\,\mathrm{signum}(-g+f+1) - 2\,\mathrm{signum}(-g+f-1) + 2\,\mathrm{signum}(-2\,g+f))\,g - 2\,\mathrm{signum}(2+f) - 2\,\mathrm{signum}(-g+f+1) + 2\,\mathrm{signum}(g+f+1) + 2\,\mathrm{signum}(g+f-1) + 2\,\mathrm{signum}(-2+f) - 2\,\mathrm{signum}(-g+f-1))\,f$$

For $g=0$ these of course reduce to much simpler expressions. The *cos* term becomes

```
G[c](0,f);
```

$$(-\mathrm{signum}(-2+f) - \mathrm{signum}(2+f) + 2\,\mathrm{signum}(f))\,f^2 + (-2\,\mathrm{signum}(2+f) + 2\,\mathrm{signum}(-2+f))\,f$$

```
factor( convert(",abs) );
```

$$(-|-2+f| - |2+f| + 2\,|f|)\,f$$

This is identical to eq. (11) in TM97-01. For $2<|f|$ this expression is identically zero. The *sin* term becomes

```
G[s](0,f);
```

$$(\mathrm{signum}(2+f) - 2\,\mathrm{signum}(f) + \mathrm{signum}(-2+f))\,f^2 + (-2\,\mathrm{signum}(-2+f) + 2\,\mathrm{signum}(2+f))\,f$$

```
factor( convert(",abs) );
```

$$(-2\,|f| + |-2+f| + |2+f|)\,f$$

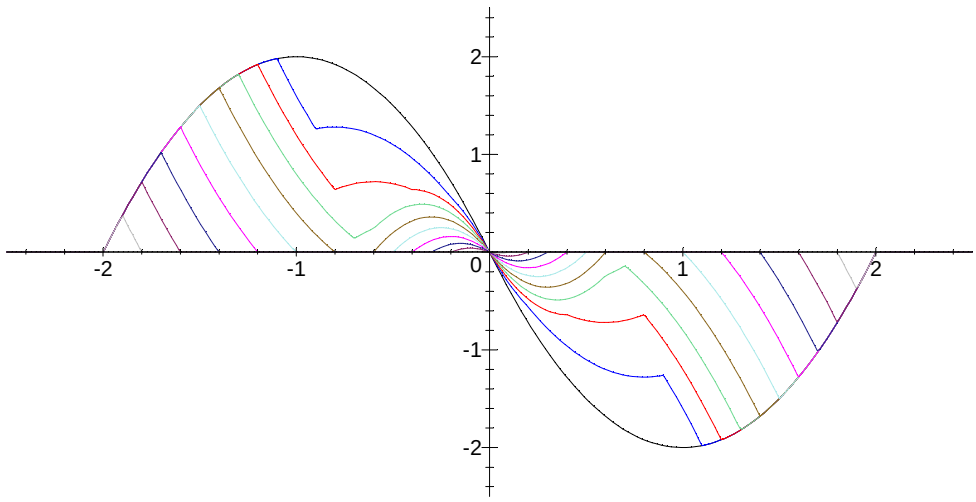
which is negative the *cos* term when $g=0$. Let's look at some plots.

```
Gplot := proc( gvals::list, frange::range, cs::name )
  local p, k, F, Grange;
  if cs=`c` then
      F := G[c];
  elif cs=`s` then
      F := G[s];
  fi;
  if nargs=4 then
    Grange := args[4];
  else
    Grange := -2.5..2.5;
  fi;
  p := [];
  for k from 1 to nops(gvals) do
    p := [ op(p), plot( F(gvals[k],f), f=frange,
                        color=mycolors[(k-1 mod 10)+1],
                        view=[frange,Grange], numpoints=200 ) ];
  od;
  plots[display](p);
end:
```

```
gvals := [seq(0.1*i,i=0..10)]:
```

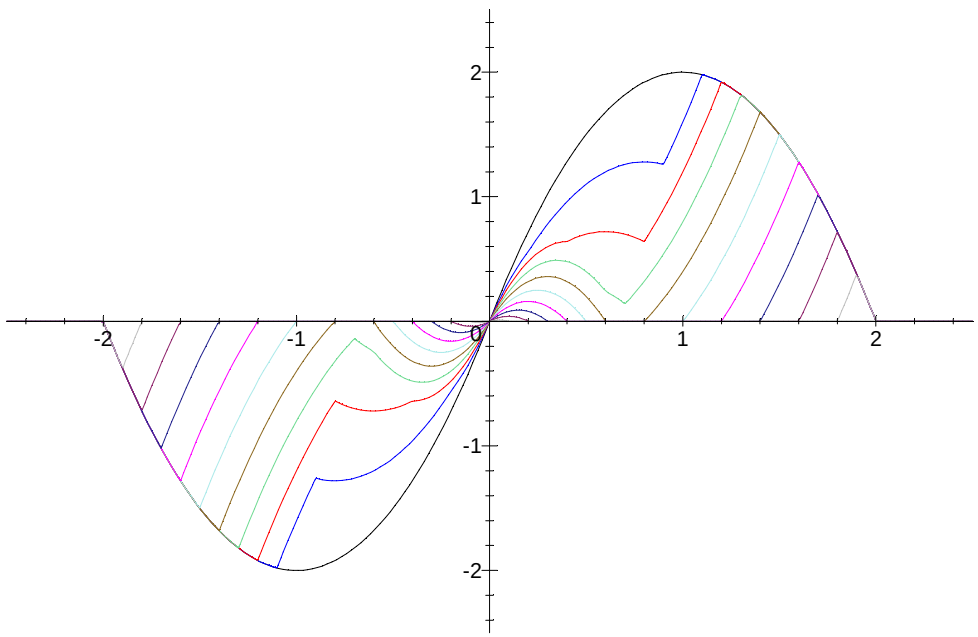
Plot the *cos* term only:

```
Gplot( gvals, -2.5..2.5, c );
```



These curves look the same as those plotted in TM97-01. Hence, the function G_c is probably identical to the *G* function in Appendix A of TM97-01. Now plot the *sin* term only:

```
Gplot( gvals, -2.5..2.5, s );
```



Oh my. So it appears that the integration done by Mathcad 5.0 in TM97-01 is incomplete. It also appears that Maple got the definite integral wrong. Finally, the sin and cos terms are negatives of each other. However, I can't quite make $G_s(g,f) + G_c(g,f) = 0$, as shown by the

following:

```
remainder := factor( convert( G[c](g,f) + G[s](g,f), abs ) );
```

$$remainder := \frac{1}{2}(-2|g-f+1|+|-2g-f|-|-2g+f|-|2g+f|-2|-g+f+1|-2|-g-f-1| -2|g+f-1|+|2g-f|+2|-1+g-f|+2|-g-f+1|+2|-g+f-1|+2|g+f+1|)f$$

Yet this goes to zero when $g=0$ and when $f=0$.

```
simplify( subs(g=0,") );
```

$$0$$

```
subs(f=0,"");
```

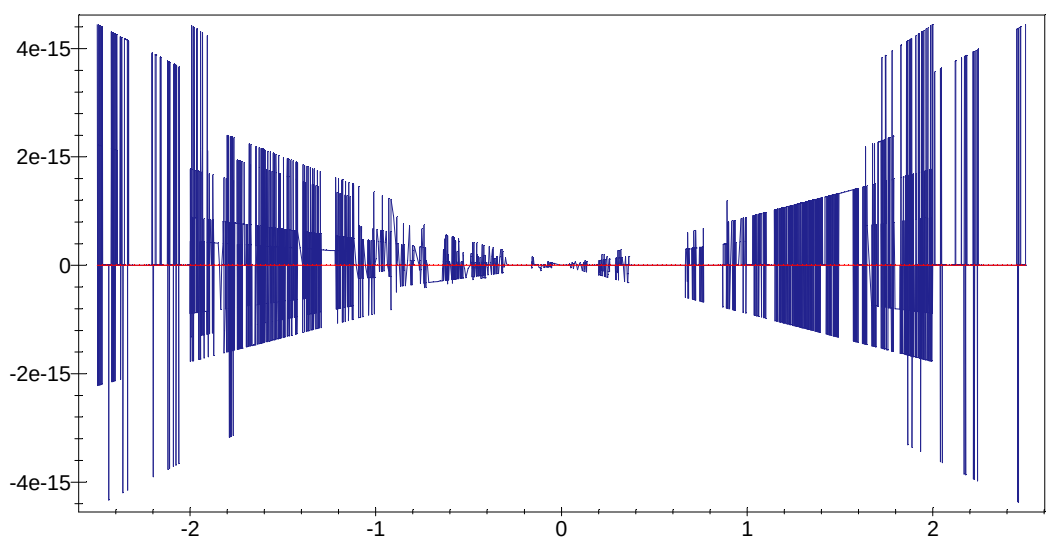
$$0$$

Let's plot $G_s(g,f)+G_c(g,f)$.

```
F := fn(remainder,g,f);
```

$$F := (g,f) \to \frac{1}{2}(-2|g-f+1|+|-2g-f|-|-2g+f|-|2g+f|-2|-g+f+1|-2|-g-f-1| -2|g+f-1|+|2g-f|+2|-1+g-f|+2|-g-f+1|+2|-g+f-1|+2|g+f+1|)f$$

```
plot( [F(0.1,f),0], f=-2.5..2.5, numpoints=200,
      axes=BOX, color=[navy,red] );
```



This is just numerical noise. Hence it does indeed appear that $G_c(g,f)=-G_s(g,f)$.