Space Curves

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restart

with(*curves*)

[*ParametrizedToImplicit*, *arclength*, *astroid*, *binormal*, *cardioid*, *cassinian*, *catenary*, *circle*, *cissoid*, *clotharg*, *clothoid*, *curvature*, *cycloid*, *deltoid*, *ellipse*, *evolute*, *folium*, *helix*, *hyperbola*, *inversion*, *involute*, *lemniscate*, *lissajous*, *logspiral*, *parabola*, *pedal*, *radial*, *tangent*, *torsion*, *torusknot*, *tractrix*, *twicubic*, *unitnormal*]

Space Curves

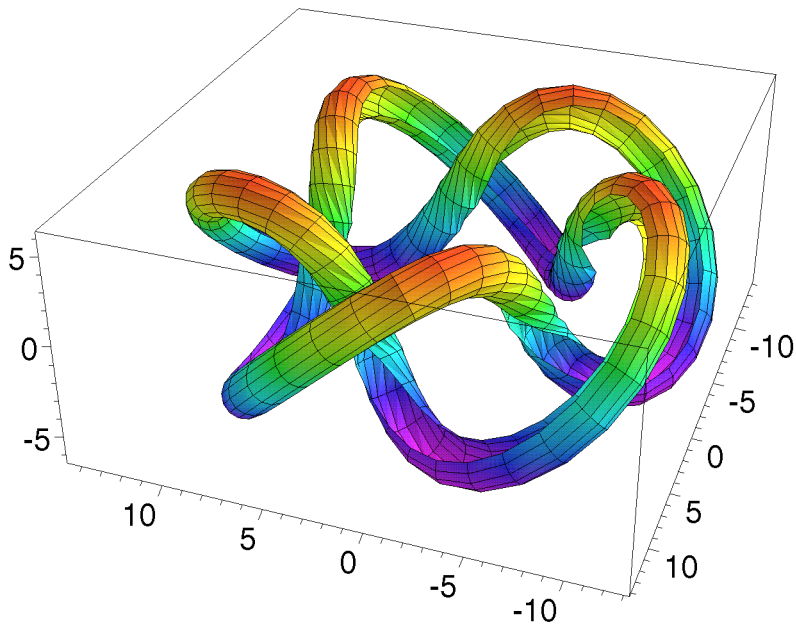
Torus Knot

$TorusKnot := (a, b, c, p, q, t) \rightarrow [(a + b\cos(q\,t))\cos(p\,t), (a + b\cos(q\,t))\sin(p\,t), c\sin(q\,t)]$

Plot (The Luge from Hell)

tubecurveplot(TorusKnot(10, 3, 5, 2, 5, t), 1.2, $t = 0 .. 2\pi$, $\theta = .5\pi .. 2.0\pi$, *axes* = *box*, *grid* = [100, 12], *orientation* = [110, 60], *title* = *Luge from Hell*)

Luge from Hell




```
jpgplot( tubecurveplot( TorusKnot(10,3,5,2,5,t), 1.2, t=0..2*Pi,
            theta=0.5*Pi..2.0*Pi, axes=box, grid=[100,12],
            orientation=[110,60], title=`Luge from Hell` ),
          `d:/dynamics/luge.jpg` );
```

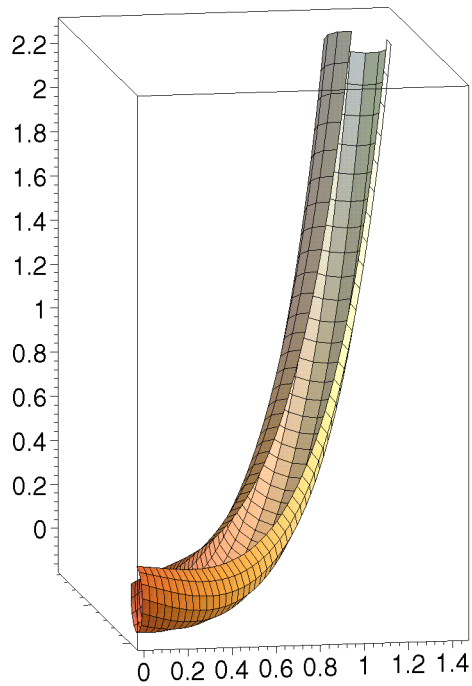
```
> resetplot();
```

Twicubic

> $Twicubic := t \rightarrow [t, t\,t, t^3]$

Plot

```
> tubecurveplot( Twicubic(t), 0.15, t=0..1.3, theta=0.5*Pi..2.1*Pi,
                 axes=box, grid=[50,12], orientation=[-100,80],
                 shading=xy, title=`Waterslide` );
```

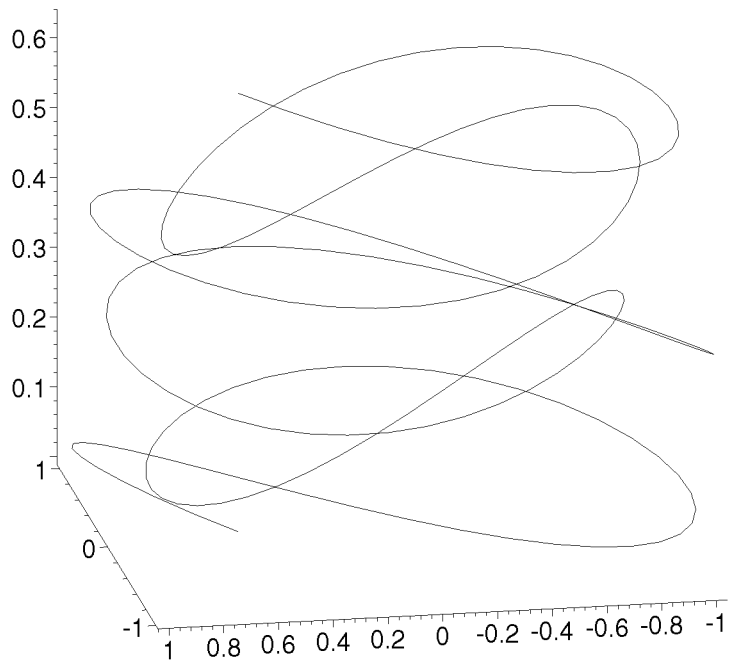


Generalized Helix (helicized plane curves)

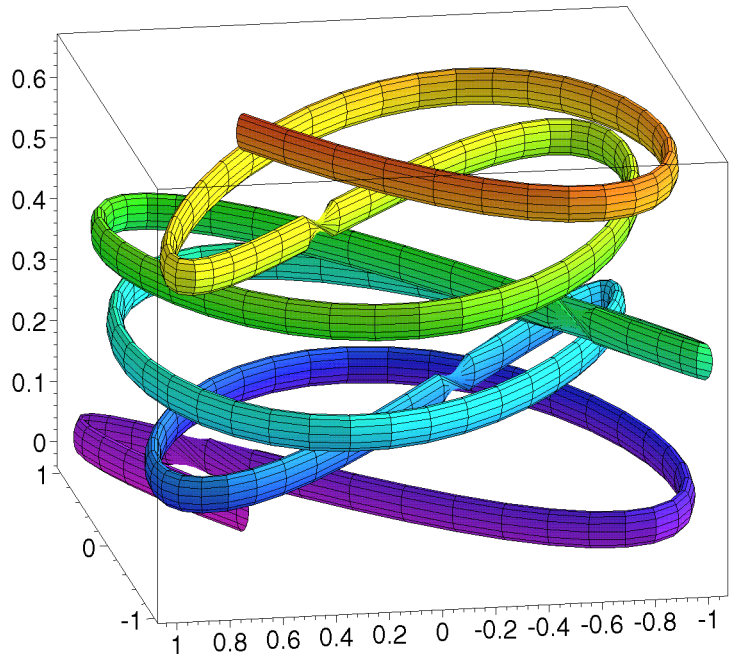
> $Helix := (curve2D, a, t) \rightarrow [\mathrm{op}(curve2D), a\ t]$

Plot of a helicized Lissajous pattern

```
> curveplot3D( Helix( Lissajous(7,5,1,1,0.6,t), 0.1, t),
               t=0..2*Pi, scaling=unconstrained,
               orientation=[170,70], shading=none, axes=frame,
               numpoints=200 );
```




```
> tubecurveplot( Helix( Lissajous(7,5,1,1,0.6,t), 0.1, t),
                 0.03, t=0..2*Pi, theta=0..2*Pi, axes=box,
                 scaling=unconstrained, grid=[200,15],
                 orientation=[170,70] );
```




```
> kappa=kappa( Helix( Lissajous(n,m,a,b,phi,t), c, t), t );
```

$$\kappa = \left(c^2 b^2 \sin(m\,t+\phi)^2 m^4 + c^2 a^2 \sin(n\,t)^2 n^4 + (-a\cos(n\,t)\,n\,b\sin(m\,t+\phi)\,m^2 + b\cos(m\,t+\phi)\,m\,a\sin(n\,t)\,n^2)^2\right)^{1/2} \Big/ \left(a^2\cos(n\,t)^2 n^2 + b^2\cos(m\,t+\phi)^2 m^2 + c^2\right)^{3/2}$$

```
> tau=tau( Helix( Lissajous(n,m,a,b,phi,t), c, t), t );
```

$$\tau = (-c\,b\sin(m\,t+\phi)\,m^2\,a\cos(n\,t)\,n^3 + c\,a\sin(n\,t)\,n^2\,b\cos(m\,t+\phi)\,m^3) \Big/ \left(c^2 b^2 \sin(m\,t+\phi)^2 m^4 + c^2 a^2 \sin(n\,t)^2 n^4 + (-a\cos(n\,t)\,n\,b\sin(m\,t+\phi)\,m^2 + b\cos(m\,t+\phi)\,m\,a\sin(n\,t)\,n^2)^2\right)$$

Junk

```
> tau=tau( Twicubic(t), t );
```

$$\tau = \frac{3}{1 + 9\,t^4 + 9\,t^2}$$

```
> kappa=kappa( Twicubic(t), t );
```

$$\kappa = 2\,\frac{\sqrt{1 + 9\,t^4 + 9\,t^2}}{(1 + 4\,t^2 + 9\,t^4)^{3/2}}$$

```
> b=Binormal( Twicubic(t), t );
```

$$b = \frac{1}{2}\,\frac{[6\,t^2, -6\,t, 2]}{\sqrt{1 + 9\,t^4 + 9\,t^2}}$$

```
> n=UnitNormal( Twicubic(t), t );
```

$$n = \frac{1}{2}\,\frac{[-4\,t - 18\,t^3, -18\,t^4 + 2, 6\,t + 12\,t^3]}{\sqrt{1 + 9\,t^4 + 9\,t^2}\,\sqrt{1 + 4\,t^2 + 9\,t^4}}$$

```
> T=Tangent( Twicubic(t), t );
```

$$T = \frac{[1, 2\,t, 3\,t^2]}{\sqrt{1 + 4\,t^2 + 9\,t^4}}$$

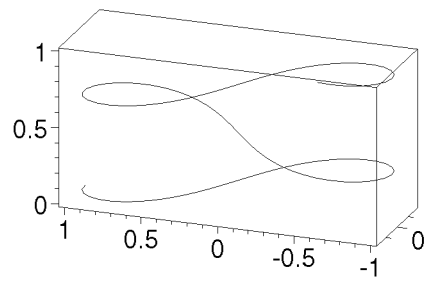
```
> foo:=linalg[crossprod]( Tangent(Twicubic(t),t),
                          UnitNormal(Twicubic(t),t) ):
> foo:=map( factor, foo );
```

$$foo := \left[3\,\frac{t^2}{\sqrt{1 + 9\,t^4 + 9\,t^2}}, -3\,\frac{t}{\sqrt{1 + 9\,t^4 + 9\,t^2}}, \frac{1}{\sqrt{1 + 9\,t^4 + 9\,t^2}}\right]$$

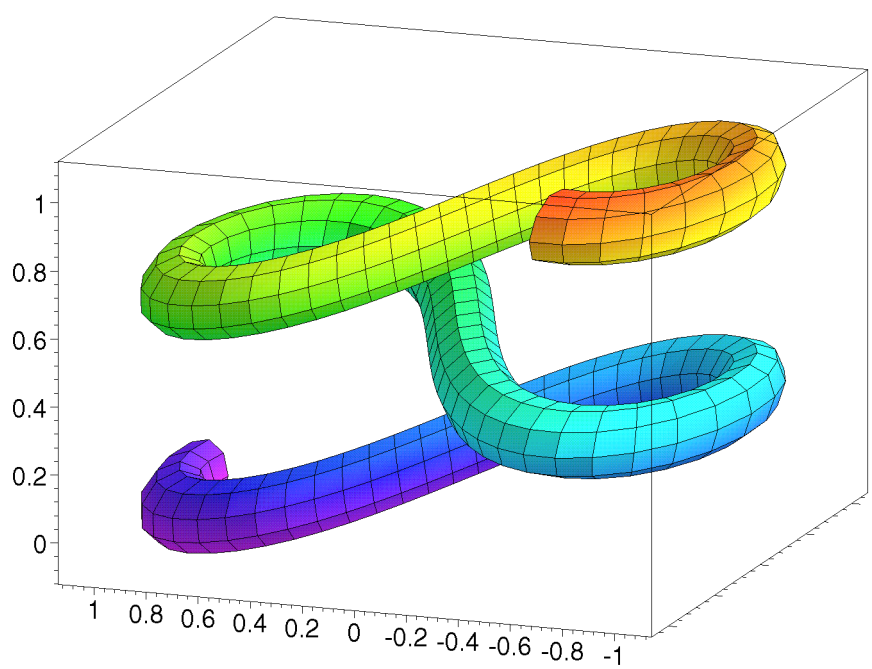
```
> evalm( foo-Binormal( Twicubic(t), t ) );
```

$$[0, 0, 0]$$

```
> curveplot3D( Helix(Lemniscate(1,t),0.1,t), t=0..10,
               shading=none, axes=box );
```




```
> tubecurveplot( Helix( Lemniscate(1,t), 0.1, t),
                 0.10, t=0..10, theta=0..2*Pi, axes=box,
                 scaling=unconstrained, grid=[100,10] );
```




```
> tau = simplify( tau( Helix( Lemniscate(a,t), c, t), t ), trig );
```

$$\tau = -6\,\frac{(\cos(t)^2 - 6)\cos(t)\,c}{c^2\cos(t)^4 - 12\,c^2\cos(t)^2 + 20\,c^2 + 9\,a^2}$$

More Junk

```
> jpgplot( tubecurveplot( Helix( Lemniscate(1,t), 0.1, t),
           0.10, t=0..10, theta=0..2*Pi, axes=box,
           scaling=unconstrained, grid=[100,10] ), `m:/foo.jpg` );
> resetplot();
> foo := kappa( TorusKnot(a,b,c,n,m,t), t ):
> foo := simptrig( simplify( expand(foo,trig), trig ), m*t );
```

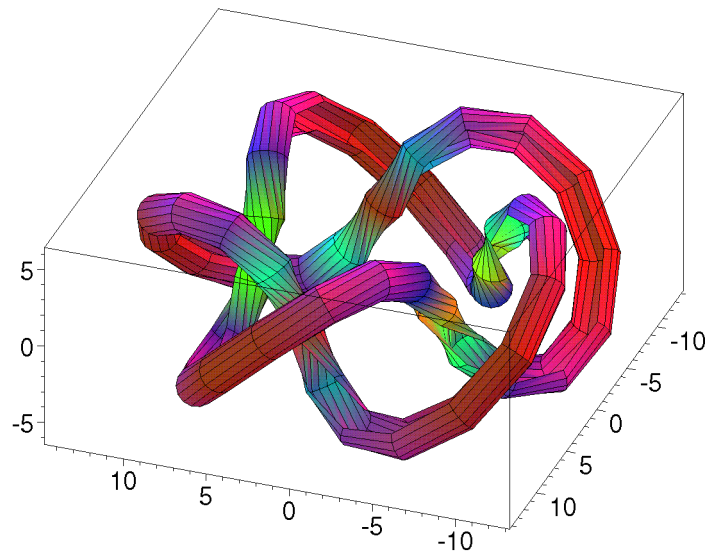
$$foo := \Big((-4\,n^6\,a\,b^3 + 2\,n^4\,a\,b^3\,m^2 + 2\,n^2\,a\,b^3\,m^4 - 2\,c^2\,m^2\,n^4\,a\,b - 2\,c^2\,m^4\,b\,n^2\,a)\cos(m\,t)\sin(m\,t)^2 + (-2\,n^4\,b^4\,m^2 - c^2\,m^4\,b^2\,n^2 + b^4\,m^4\,n^2 + n^6\,b^4 + c^2\,m^2\,n^4\,b^2)\sin(m\,t)^4 + (6\,n^4\,a\,b^3\,m^2 + 2\,n^2\,a\,b^3\,m^4 + 2\,n^4\,a^3\,b\,m^2 + 4\,n^6\,a^3\,b + 4\,n^6\,a\,b^3 + 2\,c^2\,m^2\,n^4\,a\,b + 2\,c^2\,m^4\,b\,n^2\,a)\cos(m\,t) + n^6\,a^4 + c^2\,m^6\,b^2 + 6\,n^4\,a^2\,b^2\,m^2 + b^4\,m^4\,n^2 + 2\,n^4\,b^4\,m^2 + n^2\,a^2\,b^2\,m^4 + 2\,c^2\,m^4\,b^2\,n^2 + 6\,n^6\,a^2\,b^2 + c^2\,m^2\,n^4\,a^2 + n^6\,b^4 + c^2\,m^2\,n^4\,b^2 + (-c^2\,m^4\,b^2\,n^2 + 2\,b^4\,m^4\,n^2 - 2\,n^6\,b^4 - 2\,c^2\,m^2\,n^4\,b^2 + c^2\,m^4\,n^2\,a^2 - 2\,n^4\,a^2\,b^2\,m^2 - n^2\,a^2\,b^2\,m^4 - 6\,n^6\,a^2\,b^2 - c^2\,m^2\,n^4\,a^2)\sin(m\,t)^2\Big)^{1/2} \Big/ \Big(2\,n^2\,a\,b\cos(m\,t) + n^2\,a^2 + n^2\,b^2 + c^2\,m^2 + (-n^2\,b^2 + b^2\,m^2 - c^2\,m^2)\sin(m\,t)^2\Big)^{3/2}$$

```
> kap := function( foo, a,b,c,n,m,t ):
> tor := simptrig( simplify( tau( TorusKnot(a,b,c,n,m,t), t ), trig ), m*t );
```

$$tor := \Big(-n^3\,c\,m\,b^2\,(-m^2 + n^2)\cos(m\,t)\sin(m\,t)^2 + n\,c\,m\,(2\,m^4\,b^2 - n^2\,m^2\,a^2 + 3\,b^2\,m^2\,n^2 + n^4\,a^2 + b^2\,n^4)\cos(m\,t) + n\,c\,m\,a\,b\,(2\,m^2\,n^2 - m^4 + 2\,n^4) - 2\,n^3\,c\,m\,a\,b\,(2\,m^2 + n^2)\sin(m\,t)^2\Big) \Big/ \Big((-4\,n^6\,a\,b^3 + 2\,n^4\,a\,b^3\,m^2 + 2\,n^2\,a\,b^3\,m^4 - 2\,c^2\,m^2\,n^4\,a\,b - 2\,c^2\,m^4\,b\,n^2\,a)\cos(m\,t)\sin(m\,t)^2 + (-2\,n^4\,b^4\,m^2 - c^2\,m^4\,b^2\,n^2 + b^4\,m^4\,n^2 + n^6\,b^4 + c^2\,m^2\,n^4\,b^2)\sin(m\,t)^4 + (6\,n^4\,a\,b^3\,m^2 + 2\,n^2\,a\,b^3\,m^4 + 2\,n^4\,a^3\,b\,m^2 + 4\,n^6\,a^3\,b + 4\,n^6\,a\,b^3 + 2\,c^2\,m^2\,n^4\,a\,b + 2\,c^2\,m^4\,b\,n^2\,a)\cos(m\,t) + n^6\,a^4 + c^2\,m^6\,b^2 + 6\,n^4\,a^2\,b^2\,m^2 + b^4\,m^4\,n^2 + 2\,n^4\,b^4\,m^2 + n^2\,a^2\,b^2\,m^4 + 2\,c^2\,m^4\,b^2\,n^2 + 6\,n^6\,a^2\,b^2 + c^2\,m^2\,n^4\,a^2 + n^6\,b^4 + c^2\,m^2\,n^4\,b^2 + (-c^2\,m^4\,b^2\,n^2 + 2\,b^4\,m^4\,n^2 - 2\,n^6\,b^4$$

$$-2\,c^2\,m^2\,n^4\,b^2+c^2\,m^4\,n^2\,a^2-2\,n^4\,a^2\,b^2\,m^2-n^2\,a^2\,b^2\,m^4-6\,n^6\,a^2\,b^2-c^2\,m^2\,n^4\,a^2)$$

$$\sin(m\,t)^2)$$

```
> tor := function( tor, a,b,c,n,m,t ):
```

```
> tubecurveplot( TorusKnot(10,3,5,2,5,t), 1.2, t=0..2*Pi,
                 theta=0.5*Pi..2.0*Pi, axes=box, grid=[50,12],
                 orientation=[110,60],
                 color=5*tor(10,3,5,2,5,t) );
```