Rigid Body Dynamics of a Spinning Truncated Cone Immersed in a Pressure Field

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Euler Equations of Motion for a Rigid Body

restart;
alias($I = I$, $\psi = \psi(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$, $\Omega_x = \Omega_x(t)$, $\Omega_y = \Omega_y(t)$, $\Omega_z = \Omega_z(t)$, $\Omega_\phi = \Omega_\phi(t)$, $\Omega_\psi = \Omega_\psi(t)$, $\Omega_\theta = \Omega_\theta(t)$)

In the frame of reference attached to the rotating body (the body frame), the equations of motion are

$$EulerEqs := \mathrm{mat}\left(I_x\left(\frac{\partial}{\partial t}\Omega_x\right) + (I_z - I_y)\,\Omega_y\,\Omega_z - K_x,\ I_y\left(\frac{\partial}{\partial t}\Omega_y\right) + (I_x - I_z)\,\Omega_x\,\Omega_z - K_y,\ I_z\left(\frac{\partial}{\partial t}\Omega_z\right) + (I_y - I_x)\,\Omega_x\,\Omega_y - K_z\right)$$

$$EulerEqs := \begin{bmatrix} I_x\left(\frac{\partial}{\partial t}\Omega_x\right) + (I_z - I_y)\,\Omega_y\,\Omega_z - K_x \\ I_y\left(\frac{\partial}{\partial t}\Omega_y\right) + (I_x - I_z)\,\Omega_x\,\Omega_z - K_y \\ I_z\left(\frac{\partial}{\partial t}\Omega_z\right) + (I_y - I_x)\,\Omega_x\,\Omega_y - K_z \end{bmatrix}$$

latex(*EulerEqs*, "d:/dynamics/precession/EulerEqs.tex")

where I_x, I_y, I_z are the principal moments of inertia of the body, $\Omega_x, \Omega_y, \Omega_z$ are the angular velocities of the body about the principal axes, and K_x, K_y, K_z are the components of the torque acting on the body.

Eulerian Angle Transformation

Rotation Matrix

Construct a rotation matrix that transforms coordinates from the fixed frame (X,Y,Z) to the body frame (x,y,z). First, rotate the coordinates ccw around the Z axis.

$$r1 := \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, rotate ccw around the X' axis.

$$r2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

Next, rotate ccw around the Z'' axis.

$$r3 := \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now combine the rotations into a single rotation matrix.

$$R := (p, q, r) \rightarrow \mathrm{evalm}((\mathrm{subs}(\theta = r, \mathrm{eval}(r3)) \text{ `\&*` } \mathrm{subs}(\psi = q, \mathrm{eval}(r2))) \text{ `\&*` } \mathrm{subs}(\phi = p, \mathrm{eval}(r1)))$$

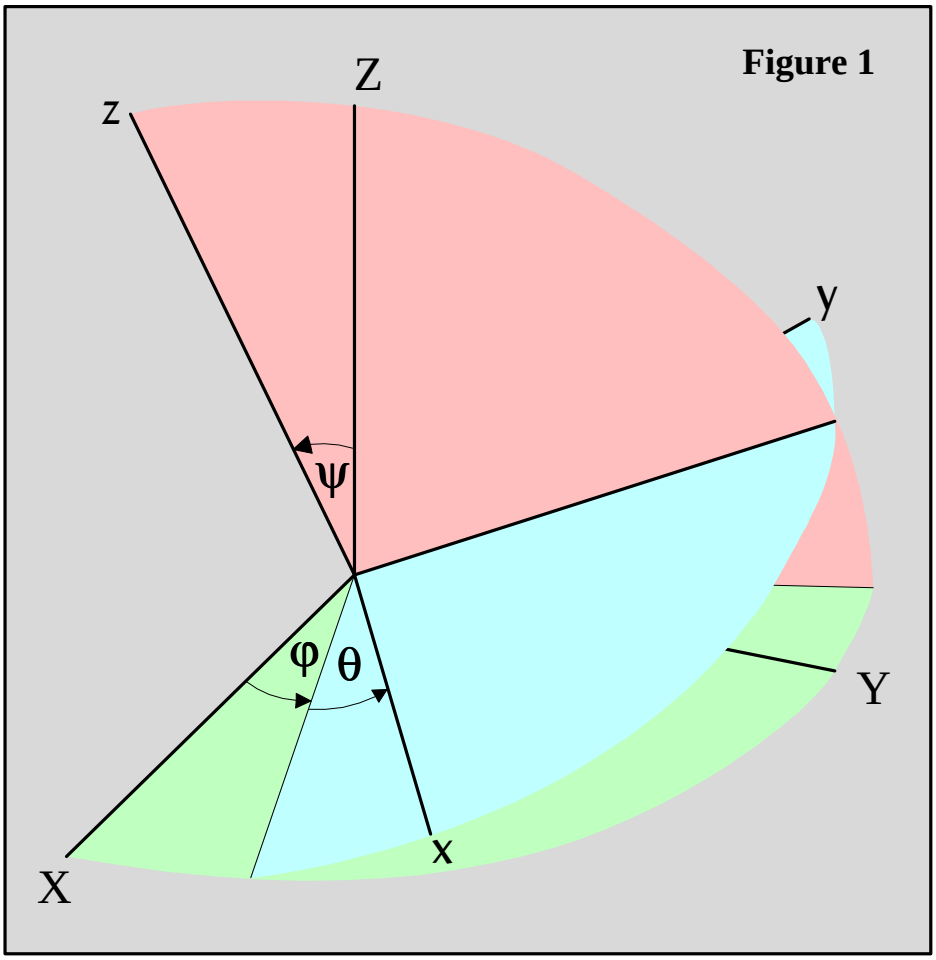
Hence, we have the coordinate transformation

$$\mathrm{mat}(x, y, z) = \mathrm{R}(\phi, \psi, \theta) \text{ \&* } \mathrm{mat}(X, Y, Z)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi), \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi), \sin(\theta)\sin(\psi)]$$
$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi), -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$
$$\cos(\theta)\sin(\psi)]$$
$$[\sin(\psi)\sin(\phi), -\sin(\psi)\cos(\phi), \cos(\psi)] \text{ \&* } \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

latex(%, "d:/dynamics/precession/FixedToBody.tex")

The diagram below illustrates the three rotations.



Angular Velocity Vector in the Body Frame

The angular velocity vector may be decomposed into components along each of the rotation axes used to construct the transformation matrix. If we transform those components to the body frame, then we can express the angular velocity vector in the body frame in terms of the Euler angles (ϕ, ψ, θ). The component of Ω along the first rotation axis, as viewed in the body frame, is

$$\omega_\phi := \mathrm{evalm}(R(0, \psi, \theta)\ \&*\ \mathrm{mat}(0, 0, 1))\left(\frac{\partial}{\partial t}\phi\right)$$

$$\omega_\phi := \begin{bmatrix} \sin(\theta)\sin(\psi) \\ \cos(\theta)\sin(\psi) \\ \cos(\psi) \end{bmatrix}\left(\frac{\partial}{\partial t}\phi\right)$$

The component along the Y' axis is, in the body frame,

$$\omega_\psi := \mathrm{evalm}(R(0, 0, \theta)\ \&*\ \mathrm{mat}(1, 0, 0))\left(\frac{\partial}{\partial t}\psi\right)$$

$$\omega_\psi := \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{bmatrix} \left(\frac{\partial}{\partial t}\psi\right)$$

Finally, the component along the Z" axis is simply

$$\omega_\theta := \mathrm{mat}(0, 0, 1)\left(\frac{\partial}{\partial t}\theta\right)$$

$$\omega_\theta := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\frac{\partial}{\partial t}\theta\right)$$

Hence, we have the angular velocity vector in the body frame,

$$\Omega = \mathrm{mat}\left(\mathrm{seq}\left(\mathrm{evalm}(\omega_\phi)_{i,1} + \mathrm{evalm}(\omega_\psi)_{i,1} + \mathrm{evalm}(\omega_\theta)_{i,1}, i = 1 .. 3\right)\right)$$

$$\Omega = \begin{bmatrix} \left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right) \end{bmatrix}$$

latex(%, "d:/dynamics/precession/OmegaBodyFrame.tex")

$Obody$:= convert(rhs(%), $vector$)

Rigid Body Equations — General Case

Equations of Motion

Substituting back into the Euler equations, we find

$xyz := [x, y, z]$

$MI := [I_x, I_y, I_z]$

$$\mathrm{subs}\left(\mathrm{seq}\left(\Omega_{xyz_k}(t) = Obody_k, k = 1 .. 3\right), \mathrm{evalm}(EulerEqs)\right)$$

$$\left[I_x\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\right) + (I_z - I_y)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right)\right) - K_x\right]$$

$$\left[I_y\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)\right)\right.$$

$$+(I_x - I_z)\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)-K_y\Bigg]$$

$$\Bigg[I_z\left(\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)\right)$$

$$+(I_y - I_x)\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi)-\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)$$

$$-K_z\Bigg]$$

$$\left[\mathrm{seq}\left(\mathrm{collect}\left(\frac{\%_{i,1}}{MI_i},\left[\frac{\partial^2}{\partial t\,\partial t}\theta,\frac{\partial^2}{\partial t\,\partial t}\psi,\frac{\partial^2}{\partial t\,\partial t}\phi,\sin(\psi),\frac{\partial}{\partial t}\psi,\sin(\theta),\cos(\theta),\mathit{diff}\right],\mathit{simplify}\right),i=1\,..\,3\right)\right]$$

$$\Bigg[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)$$

$$+\left(\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}+\frac{(I_z-I_y)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_x}\right)\cos(\theta)\sin(\psi)$$

$$+\left(-\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}+\frac{\cos(\psi)\,(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_x}{I_x},-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)$$

$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)$$

$$+\left(\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}+\frac{(I_x-I_z)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_y}\right)\sin(\theta)\sin(\psi)$$

$$+\left(\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}+\frac{\cos(\psi)\,(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_y}{I_y},\left(\frac{\partial^2}{\partial t^2}\theta\right)$$

$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\theta)\sin(\theta)\sin(\psi)^2}{I_z}$$

$$+\left(\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)^2}{I_z}-\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)^2}{I_z}-\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\psi)$$

$$+\frac{(-I_y+I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)^2}{I_z}-\frac{K_z}{I_z}\Bigg]$$

$tmp := \%$

Clean up the third equation a bit

$$\text{algsubs}(\cos(\theta)\sin(\theta)=A,\ tmp_3)$$

$$\text{algsubs}\left(\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)=\frac{B}{\frac{\partial}{\partial t}\phi},\ \%\right)$$

$$\text{algsubs}(I_x-I_y=C\,I_z,\ \%)$$

$$\left(\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)^2}{I_z}-\frac{(-I_y+I_x)\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)^2}{I_z}-\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\psi)$$
$$+\frac{(-I_y+I_x)\,A\left(\left(\frac{\partial}{\partial t}\psi\right)^2-\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2\right)}{I_z}+\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{K_z}{I_z}$$

$$A\left(\left(\frac{\partial}{\partial t}\psi\right)^2-\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2\right)C-\frac{B\,(\cos(\theta)^2\,C\,I_z-\sin(\theta)^2\,C\,I_z+I_z)}{I_z}+\left(\frac{\partial^2}{\partial t^2}\theta\right)$$
$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{K_z}{I_z}$$

$$\text{Collect}\left(\%,\left[\frac{\partial^2}{\partial t\,\partial t}\theta,\frac{\partial^2}{\partial t\,\partial t}\phi,A,B,C,\left(\frac{\partial}{\partial t}\phi\right)^2,\cos(\theta)\right],\ simplify,\ alg\right)$$

$$\text{subs}\left(A=\cos(\theta)\sin(\theta),\ B=\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right),\ C=\frac{I_x-I_y}{I_z},\ \%\right)$$

$$\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)+\frac{\left((-1+\cos(\psi)^2)\left(\frac{\partial}{\partial t}\phi\right)^2+\left(\frac{\partial}{\partial t}\psi\right)^2\right)(-I_y+I_x)\cos(\theta)\sin(\theta)}{I_z}$$
$$+\left(\frac{(-2\cos(\theta)^2+1)\,(-I_y+I_x)}{I_z}-1\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_z}{I_z}$$

$tmp_3 := \%$

$BodyEqs := \mathrm{eval}(tmp)$

latex($BodyEqs$, "d:/dynamics/precession/RigidBodyEqs.tex")

mat($BodyEqs$)

$$\left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}+\frac{(I_z-I_y)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_x}\right)\cos(\theta)\sin(\psi)+\left(-\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}+\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_x}{I_x}\right]$$

$$\left[-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}+\frac{(I_x-I_z)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_y}\right)\sin(\theta)\sin(\psi)+\left(\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}+\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_y}{I_y}\right]$$

$$\left[\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)+\frac{\left((-1+\cos(\psi)^2)\left(\frac{\partial}{\partial t}\phi\right)^2+\left(\frac{\partial}{\partial t}\psi\right)^2\right)(-I_y+I_x)\cos(\theta)\sin(\theta)}{I_z}+\left(\frac{(-2\cos(\theta)^2+1)(-I_y+I_x)}{I_z}-1\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_z}{I_z}\right]$$

Rigid Body Equations — Symmetric Top

General Motion of a Symmetric Top

Equations of Motion

$$SymTop := \mathrm{subs}(I_x = I_y, I_y = I_{xy}, BodyEqs)$$

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{I_z\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}} + \frac{(I_z - I_{xy})\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_{xy}}\right)\cos(\theta)\sin(\psi) + \left(-\frac{I_z\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}} + \frac{\cos(\psi)(2I_{xy} - I_z)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_x}{I_{xy}}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left(-\frac{I_z\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}} + \frac{(I_{xy} - I_z)\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)}{I_{xy}}\right)\sin(\theta)\sin(\psi) + \left(-\frac{I_z\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}} + \frac{\cos(\psi)(2I_{xy} - I_z)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\cos(\theta)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_y}{I_{xy}}, \left(\frac{\partial^2}{\partial t^2}\theta\right) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) - \left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right) - \frac{K_z}{I_z}\right]$$

$$SymTop := \left[\mathrm{seq}\left(\mathrm{collect}\left(\mathrm{simplify}\left(SymTop_k, \left\{1 - \frac{I_z}{I_{xy}} = \beta\right\}\right), [I_z]\right), \left[\frac{\partial^2}{\partial t\,\partial t}\psi, \frac{\partial^2}{\partial t\,\partial t}\phi, \sin(\theta), \cos(\theta), \sin(\psi), \frac{\partial}{\partial t}\psi, diff\right], factor\right), k = 1 .. 3\right)\right]$$

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left((-1+\beta)\left(\frac{\partial}{\partial t}\theta\right) + \cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) + \left((1-\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right) - \left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\beta\right)\sin(\psi)\cos(\theta) - \frac{K_x}{I_{xy}}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left((-1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right) + \left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\beta\right)\sin(\psi)\sin(\theta)$$

$$+\left((-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)+\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)-\frac{K_y}{I_{xy}},$$

$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)+\frac{K_z}{I_{xy}(-1+\beta)}\Bigg]$$

latex($SymTop_1$, "d:/dynamics/precession/SymTop1.tex");

latex($SymTop_2$, "d:/dynamics/precession/SymTop2.tex");

latex($SymTop_3$, "d:/dynamics/precession/SymTop3.tex")

Conversion to a System of First-Order ODEs

We may write the equations of motion as a system of first-order differential equations.

$$subslist := \left[\frac{\partial}{\partial t}\phi=\Omega_\phi, \frac{\partial}{\partial t}\psi=\Omega_\psi, \frac{\partial}{\partial t}\theta=\Omega_\theta\right]$$

subs($subslist$, $SymTop$)

$$\text{solve}\left(\{\text{op}(\%)\}, \{\frac{\partial}{\partial t}\Omega_\phi, \frac{\partial}{\partial t}\Omega_\psi, \frac{\partial}{\partial t}\Omega_\theta\}\right)$$

collect(%, [I_{xy}, Ω_ψ, $\sin(\psi)$, Ω_θ], $factor$)

$$\Bigg\{\frac{\partial}{\partial t}\Omega_\psi=\left((-1+\beta)\Omega_\phi\Omega_\theta+\Omega_\phi^{\,2}\cos(\psi)\beta\right)\sin(\psi)+\frac{-K_y\sin(\theta)+\cos(\theta)K_x}{I_{xy}},$$

$$\frac{\partial}{\partial t}\Omega_\phi=\frac{((1-\beta)\Omega_\theta-\cos(\psi)(1+\beta)\Omega_\phi)\Omega_\psi}{\sin(\psi)}+\frac{\cos(\theta)K_y+K_x\sin(\theta)}{\sin(\psi)I_{xy}}, \frac{\partial}{\partial t}\Omega_\theta=$$

$$\left(-\Omega_\phi\beta\sin(\psi)+\frac{(-1+\beta)\cos(\psi)\Omega_\theta+(1+\beta)\Omega_\phi}{\sin(\psi)}\right)\Omega_\psi$$

$$+\frac{-\frac{K_z}{-1+\beta}-\frac{(\cos(\theta)K_y+K_x\sin(\theta))\cos(\psi)}{\sin(\psi)}}{I_{xy}}\Bigg\}$$

$foo := \%$

$$\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\phi\right)$$

collect(op(%) $\sin(\psi)$, [I_{xy}, Ω_ψ, Ω_θ], $factor$)

foo := (foo minus %%) union {%}

$$\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\theta\right)$$

$$\text{collect}(\text{sinfix}(\text{op}(\%)\sin(\psi), \psi), [I_{xy}, \Omega_\psi, \Omega_\phi, K_z], factor)$$

$$foo := (foo \text{ minus } \%\%) \text{ union } \{\%\}$$

$$foo := \left\{\frac{\partial}{\partial t}\Omega_\psi = \left((-1+\beta)\,\Omega_\phi\,\Omega_\theta + \Omega_\phi^{\,2}\cos(\psi)\,\beta\right)\sin(\psi) + \frac{-K_y\sin(\theta) + \cos(\theta)\,K_x}{I_{xy}},\right.$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{\cos(\theta)\,K_y + K_x\sin(\theta)}{I_{xy}},$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi$$

$$\left. + \frac{-\dfrac{\sin(\psi)\,K_z}{-1+\beta} - (\cos(\theta)\,K_y + K_x\sin(\theta))\cos(\psi)}{I_{xy}}\right\}$$

$$FirstOrderODEsK := \left[\frac{\partial}{\partial t}\phi = \Omega_\phi, \frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\theta = \Omega_\theta, \text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\phi\right)\right),\right.$$
$$\left.\text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\psi\right)\right), \text{op}\left(\text{select}\left(has, foo, \frac{\partial}{\partial t}\Omega_\theta\right)\right)\right]$$

$$\text{mat}(FirstOrderODEsK)$$

$$\left[\frac{\partial}{\partial t}\phi = \Omega_\phi\right]$$

$$\left[\frac{\partial}{\partial t}\psi = \Omega_\psi\right]$$

$$\left[\frac{\partial}{\partial t}\theta = \Omega_\theta\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{\cos(\theta)\,K_y + K_x\sin(\theta)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\psi = \left((-1+\beta)\,\Omega_\phi\,\Omega_\theta + \Omega_\phi^{\,2}\cos(\psi)\,\beta\right)\sin(\psi) + \frac{-K_y\sin(\theta) + \cos(\theta)\,K_x}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi\right.$$

$$+\frac{-\frac{\sin(\psi)\,K_z}{-1+\beta}-(\cos(\theta)\,K_y+K_x\sin(\theta))\cos(\psi)}{I_{xy}}\Bigg]$$

latex(%, "d:/dynamics/precession/FirstOrderODEsK.tex")

Force-Free Motion of a Symmetric Top

$FFSymTop := \mathrm{subs}(K_x = 0, K_y = 0, K_z = 0, SymTop)$

$$FFSymTop := \Bigg[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)$$
$$+\left((-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)+\cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)$$
$$+\left((1-\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)-\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\,\beta\right)\sin(\psi)\cos(\theta),\ -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)$$
$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left((-1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right)+\left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\,\beta\right)\sin(\psi)\sin(\theta)$$
$$+\left((-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)+\cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\right)\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta),$$
$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)\Bigg]$$

We notice that the third equation,

$FFSymTop_3$

$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)$$

can be written as

$$FFSymTop_3 = \frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)$$
$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\frac{\partial^2}{\partial t^2}\theta\right)=\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)$$

rhs(%) − lhs(%)

$$0$$

Hence, $\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)=const$

This is the projection of the angular velocity onto the symmetry axis. Recall the components of Ω in the body frame:

$\mathrm{mat}(\Omega_x, \Omega_y, \Omega_z) = \mathrm{convert}(Obody, matrix)$

$$\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix}=\begin{bmatrix}\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi)-\left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\end{bmatrix}$$

We see that the angular velocity about the symmetry axis, Ω_z, is a constant of the motion. We also have conservation of energy,

$$E=\frac{I_x\Omega_x^2+I_y\Omega_y^2+I_z\Omega_z^2}{2}$$

$KE := \%$

Substituting the components of Ω in terms of the Euler angles, we have

$\mathrm{collect}\left(\mathrm{subs}\left(\mathrm{seq}\left(\Omega_{xyz_k}(t)=Obody_k, k=1..3\right), KE\right), [diff, \sin(\psi)], factor\right)$

$KE := \%$

$$E=\frac{1}{2}I_z\left(\frac{\partial}{\partial t}\theta\right)^2+I_z\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)\left(\frac{\partial}{\partial t}\theta\right)$$
$$+\left(\left(\frac{1}{2}I_x\sin(\theta)^2+\frac{1}{2}I_y\cos(\theta)^2\right)\sin(\psi)^2+\frac{1}{2}I_z\cos(\psi)^2\right)\left(\frac{\partial}{\partial t}\phi\right)^2$$
$$+\cos(\theta)\sin(\theta)(-I_y+I_x)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+\left(\frac{1}{2}I_x\cos(\theta)^2+\frac{1}{2}I_y\sin(\theta)^2\right)\left(\frac{\partial}{\partial t}\psi\right)^2$$

$subslist := \left[\frac{\partial}{\partial t}\psi=\Omega_\psi, \frac{\partial}{\partial t}\phi=\Omega_\phi, \frac{\partial}{\partial t}\theta=\Omega_\theta\right]$

$\mathrm{subs}(subslist, I_x=I_y, I_y=I_{xy}, KE)$

$$E=\frac{1}{2}I_z\Omega_\theta^2+I_z\Omega_\phi\cos(\psi)\Omega_\theta$$
$$+\left(\left(\frac{1}{2}I_{xy}\sin(\theta)^2+\frac{1}{2}I_{xy}\cos(\theta)^2\right)\sin(\psi)^2+\frac{1}{2}I_z\cos(\psi)^2\right)\Omega_\phi^2$$
$$+\left(\frac{1}{2}I_{xy}\sin(\theta)^2+\frac{1}{2}I_{xy}\cos(\theta)^2\right)\Omega_\psi^2$$

$$\text{collect}\left(\text{simplify}\left(\frac{\%}{I_{xy}}\right), [\Omega_\phi, \Omega_\psi, \Omega_\theta, I_{xy}], \textit{simplify}\right)$$

$$\frac{E}{I_{xy}} = \left(\frac{1}{2} - \frac{1}{2}\cos(\psi)^2 + \frac{1}{2}\frac{I_z \cos(\psi)^2}{I_{xy}}\right)\Omega_\phi^{\,2} + \frac{I_z \cos(\psi)\,\Omega_\theta\,\Omega_\phi}{I_{xy}} + \frac{1}{2}\Omega_\psi^{\,2} + \frac{1}{2}\frac{I_z\,\Omega_\theta^{\,2}}{I_{xy}}$$

$$\text{collect}(\text{algsubs}(I_z = I_{xy}(1-\beta), \%), [\Omega], \textit{factor})$$

$$\frac{E}{I_{xy}} = \Omega_\phi^{\,2}\left(\frac{1}{2} - \frac{1}{2}\cos(\psi)^2\beta\right) - (-1+\beta)\cos(\psi)\,\Omega_\theta\,\Omega_\phi + \frac{1}{2}\Omega_\psi^{\,2} + \left(\frac{1}{2} - \frac{1}{2}\beta\right)\Omega_\theta^{\,2}$$

$$KE := \%$$

Steady Precession Solution

Suppose we look for a solution such that ψ is constant. Then we have

$$\text{eval}\left(\text{subs}\left(\frac{\partial}{\partial t}\psi = 0, \textit{FFSymTop}\right)\right)$$

$$\left[\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left((1-\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right) - \left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\beta\right)\sin(\psi)\cos(\theta),\right.$$
$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi) + \left((-1+\beta)\left(\frac{\partial}{\partial t}\phi\right)\left(\frac{\partial}{\partial t}\theta\right) + \left(\frac{\partial}{\partial t}\phi\right)^2\cos(\psi)\beta\right)\sin(\psi)\sin(\theta),$$
$$\left.\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) + \left(\frac{\partial^2}{\partial t^2}\theta\right)\right]$$

The first two equations can be combined to yield

$$\text{collect}\left(\frac{\%_1\sin(\theta) + \%_2\cos(\theta)}{\sin(\psi)}, [\textit{diff}], \textit{simplify}\right) = 0$$

$$\frac{\partial^2}{\partial t^2}\phi = 0$$

Hence, solutions with $\psi = \textit{const}$ force a constant precession, $\frac{\partial}{\partial t}\phi = \textit{const}$. From the third equation, we then see that $\frac{\partial}{\partial t}\theta = \textit{const}$ as well.

$$\text{subs}\left(\frac{\partial^2}{\partial t\,\partial t}\phi = 0, \%\%_1\right)$$

$$\text{solve}\left(\%, \frac{\partial}{\partial t}\phi\right)$$

$$\frac{\partial}{\partial t}\phi = \%_2$$

$$\frac{\partial}{\partial t}\phi = -\frac{\left(\frac{\partial}{\partial t}\theta\right)(-1+\beta)}{\cos(\psi)\,\beta}$$

$$\text{normal}\left(\text{subs}\left(\beta = \frac{I_{xy} - I_z}{I_{xy}}, \%\right)\right)$$

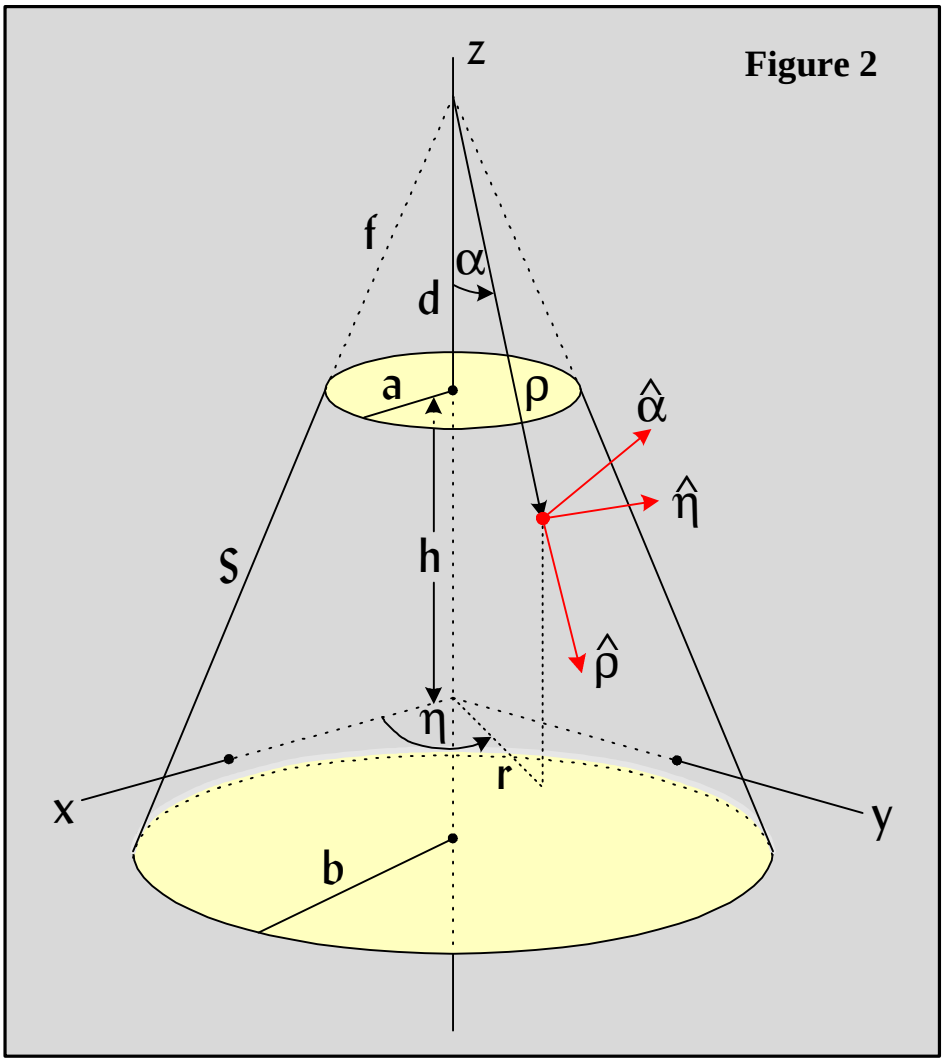
$$\frac{\partial}{\partial t}\phi = -\frac{\left(\frac{\partial}{\partial t}\theta\right)I_z}{\cos(\psi)\,(-I_{xy} + I_z)}$$

A Spinning Truncated Cone Embedded in a Pressure Field

Consider a truncated cone with small and large radii *a* and *b* and cone opening angle α. Let the center of mass reside on the (local) z axis, a distance *h* below the smaller flat surface (the top surface of the truncated cone). Then the equation for the conical surface is $\tan\alpha = \frac{r-a}{h-z}$, or $x^2 + y^2 - (a + (h-z)\tan(\alpha))^2 = 0$.

latex(%, "d:/dynamics/precession/ConeEquation.tex")

Conical Coordinates



Transformation matrix from conical to Cartesian coordinates:

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix}$$

For later convenience, make this and its inverse into functions.

$M := \%$

$ConToCart := (a, b) \rightarrow \mathrm{evalm}(\mathrm{subs}(\alpha = a, \eta = b, \mathrm{eval}(M)))$

$CartToCon := (a, b) \rightarrow \mathrm{map}(simplify, \mathrm{inverse}(\mathrm{ConToCart}(a, b)))$

latex(ConToCart(α, η), "d:/dynamics/precession/ConicalToCartesian.tex")

latex(CartToCon(α, η), "d:/dynamics/precession/CartesianToConical.tex")

Hence,

mat(*xhat*, *yhat*, *zhat*) = ConToCart(α, η) &* mat(*rhohat*, *etahat*, *alphahat*)

$$\begin{bmatrix} xhat \\ yhat \\ zhat \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} rhohat \\ etahat \\ alphahat \end{bmatrix}$$

and

mat(*rhohat*, *etahat*, *alphahat*) = CartToCon(α, η) &* mat(*xhat*, *yhat*, *zhat*)

$$\begin{bmatrix} rhohat \\ etahat \\ alphahat \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} xhat \\ yhat \\ zhat \end{bmatrix}$$

Radiation Pressure

Let $[P_X, P_Y, P_Z]$ be the pressure vector in the fixed frame. Then the conical coordinates representation of P is

mat(P_ρ, P_η, P_α) = (CartToCon(α, η) &* R(ϕ, ψ, θ)) &* mat(P_X, P_Y, P_Z)

$$\begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} = \left(\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \right.$$

$$[\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi), \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi), \sin(\theta)\sin(\psi)]$$

$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi), -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$

$$\cos(\theta)\sin(\psi)]$$

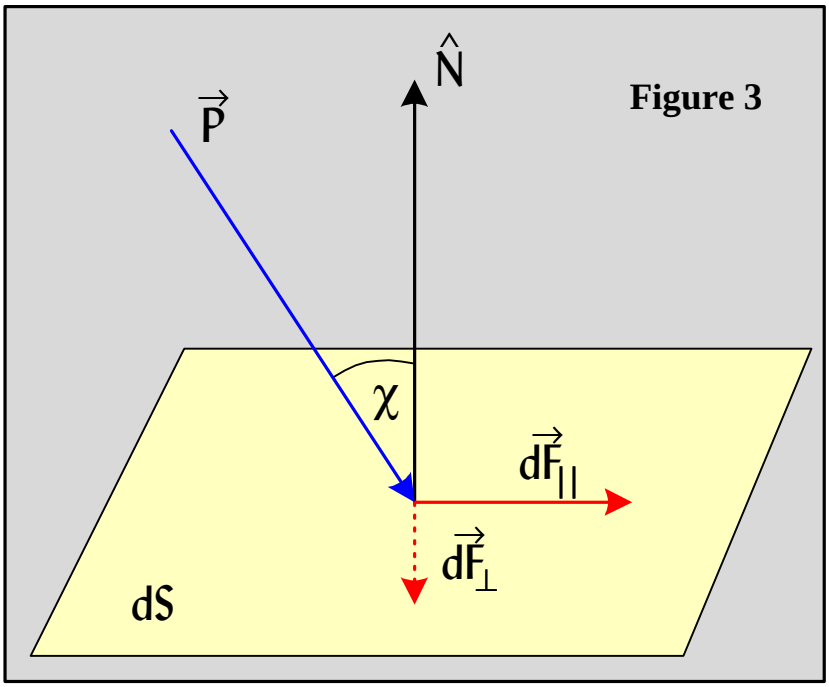
$$\left. [\sin(\psi)\sin(\phi), -\sin(\psi)\cos(\phi), \cos(\psi)] \right) \&* \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix}$$

latex(%, "d:/dynamics/precession/PressureBodyFrame.tex")

Pbody := convert(evalm(rhs(%)), *vector*)

simplify(mag(%))

$$\sqrt{P_Y^{\,2} + P_X^{\,2} + P_Z^{\,2}}$$



The force on an area element dS is

$$\mathrm{mat}(dF_{perp}, dF_{parallel}) = |P|\, dS\, |\cos(\gamma)|\, \mathrm{mat}((1 + A_C)\cos(\gamma), (1 - A_C)\sin(\gamma))$$

$$\begin{bmatrix} dF_{perp} \\ dF_{parallel} \end{bmatrix} = |P|\, dS\, |\cos(\gamma)| \begin{bmatrix} (1 + A_C)\cos(\gamma) \\ (1 - A_C)\sin(\gamma) \end{bmatrix}$$

where F_{perp} and $F_{parallel}$ are the component perpendicular and parallel to dS, γ is the angle between P and the (unit) surface normal N, and A_C is the albedo of the cone surface. From Figure 3, we see that $\mathrm{dot}(P, N) < 0$ for our problem. Hence, $\cos\gamma = -\cos\chi$ (that is, $\cos\chi = -\frac{\mathrm{dot}(P, N)}{|P|}$) and we have

$$\mathrm{mat}(dF_{perp}, dF_{parallel}) = |P|\, dS \cos(\chi)\, \mathrm{mat}(-(1 + A_C)\cos(\chi), (1 - A_C)\sin(\chi))$$

$$\begin{bmatrix} dF_{perp} \\ dF_{parallel} \end{bmatrix} = |P|\, dS \cos(\chi) \begin{bmatrix} -(1 + A_C)\cos(\chi) \\ (1 - A_C)\sin(\chi) \end{bmatrix}$$

The magnitude of dF is

$dFpp := \mathrm{convert}(\mathrm{evalm}(\mathrm{rhs}(\%)), vector)$

$\mathrm{simplify}(\sqrt{\mathrm{dot}(\%, \%)}, assume = real)$

$|dF| = \mathrm{rootfunc}(\%, collect, \cos, factor)$

$$|dF| = \mathrm{signum}(P)\, P\, \mathrm{signum}(dS)\, dS\, \mathrm{signum}(\cos(\chi))\cos(\chi)\sqrt{4\cos(\chi)^2 A_C + (A_C - 1)^2}$$

```
for p in select(has, indets(%), signum) do subs(p = 1, %) od
subs(P = |P|, %)
mag_dF := rhs(%)
```

$$|dF| = |P|\, dS \cos(\chi) \sqrt{4 \cos(\chi)^2 A_C + (A_C - 1)^2}$$

Now, $\cos\gamma = \frac{P_\alpha}{|P|}$. Hence, $\cos\chi = -\frac{P_\alpha}{|P|}$ or

```
cos χ = subs(P_X = π_X, P_Y = π_Y, P_Z = π_Z, -Pbody_3)
cos_chi := rhs(%)
latex(%, "d:/dynamics/precession/cos_chi.tex")
```

$$\cos\chi = -(\cos(\alpha)\cos(\eta)(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi)) + \sin(\alpha)\sin(\psi)\sin(\phi))\,\pi_X - (\cos(\alpha)\cos(\eta)(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi)) - \sin(\alpha)\sin(\psi)\cos(\phi))\,\pi_Y - (\cos(\alpha)\cos(\eta)\sin(\theta)\sin(\psi) + \cos(\alpha)\sin(\eta)\cos(\theta)\sin(\psi) + \sin(\alpha)\cos(\psi))\,\pi_Z$$

where $\pi_X = \frac{P_X}{|P|}$, $\pi_Y = \frac{P_Y}{|P|}$, $\pi_Z = \frac{P_Z}{|P|}$. $dF_{parallel}$ can be written $dF_{parallel} = \frac{|dF_{parallel}|\,(P - \mathrm{dot}(P, N)\,N)}{|P - \mathrm{dot}(P, N)\,N|}$. But $|P - \mathrm{dot}(P, N)\,N| = |\mathrm{cross}(P, N)|$ or $|P - \mathrm{dot}(P, N)\,N| = |P|\sin(\chi)$, and $\frac{\mathrm{dot}(P, N)}{|P|} = -\cos(\chi)$, so we can write

$$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{P + |P|\cos(\chi)\,N}{|P|\sin(\chi)}$$

$$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{P + |P|\cos(\chi)\,N}{|P|\sin(\chi)}$$

In component form,

```
subs(|P| = Q, P = mat(P_ρ, P_η, P_α), Q = |P|, N = mat(0, 0, 1), %)
```

$$\frac{dF_{parallel}}{|dF_{parallel}|} = \frac{\begin{bmatrix} P_\rho \\ P_\eta \\ P_\alpha \end{bmatrix} + |P| \cos(\chi) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{|P| \sin(\chi)}$$

For dF_{perp}, we have $\frac{dF_{perp}}{|dF_{perp}|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Therefore,

mat(dF_ρ, dF_η, dF_α) = evalm($dFpp_2$ rhs(%%) + $dFpp_1$ rhs(%))

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_\alpha \end{bmatrix} = \begin{bmatrix} dS \cos(\chi)\,(1 - A_C)\,P_\rho \\ dS \cos(\chi)\,(1 - A_C)\,P_\eta \\ dS \cos(\chi)\,(1 - A_C)\,(P_\alpha + |P| \cos(\chi)) - |P|\, dS \cos(\chi)^2\,(1 + A_C) \end{bmatrix}$$

The component perpendicular to the surface simplifies:

```
factormat := proc(M::anything)
local S, N, Q, r, c, k;
    if type(M, { `=`, `+`, `*`, list, set }) then RETURN(map(procname, M, args[2 .. nargs]))
    elif type(M, array) then
        S := { };
        if type(M, vector) then for r to size(M) do S := S union { M[r] } od
        else for r to rows(M) do for c to cols(M) do S := S union { M[r, c] } od od
        fi;
        S := S minus { 0 };
        S := map(factor, S);
        Q := { op(S[1]) };
        for k from 2 to nops(S) do Q := Q intersect { op(S[k]) } od;
        Q := convert(Q minus { -1 }, `*`);
        N := copy(M);
        N := map((x, y) -> x / y, N, Q);
        if 1 < nargs then
            N := map(args[2], N, args[3 .. nargs]); Q := map(args[2], Q, args[3 .. nargs])
        fi;
        Q*eval(N)
    else M
    fi
end
```

factormat(subs($P_\rho = \pi_\rho |P|$, $P_\eta = \pi_\eta |P|$, $P_\alpha = \pi_\alpha |P|$, %%), *collect*, [$dS$, cos], *factor*)

$$\begin{bmatrix} dF_\rho \\ dF_\eta \\ dF_\alpha \end{bmatrix} = |P| \cos(\chi)\, dS \begin{bmatrix} -(A_C - 1)\,\pi_\rho \\ -(A_C - 1)\,\pi_\eta \\ -2\,A_C \cos(\chi) - (A_C - 1)\,\pi_\alpha \end{bmatrix}$$

Thus, the force integrated over the cone surface is

$$\mathrm{mat}(F_\rho, F_\eta, F_\alpha) = \int_0^{2\pi} \int_f^{f+S} \mathrm{subs}(dS = \rho \sin(\alpha), \mathrm{rhs}(\%))\, d\rho\, d\eta$$

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} |P| \cos(\chi)\, \rho \sin(\alpha) \begin{bmatrix} -(A_C - 1)\,\pi_\rho \\ -(A_C - 1)\,\pi_\eta \\ -2\,A_C \cos(\chi) - (A_C - 1)\,\pi_\alpha \end{bmatrix} d\rho\, d\eta$$

Fintegral := %

latex(%, "d:/dynamics/precession/ConeForceIntegral.tex")

Torque Due to Radiation Pressure on the Cone Surface

Radius Vector to Cone Surface

To calculate the torque, we first need the radius vector to a point on the surface of the cone. The transformation from conical to cartesian body coordinates is

$$\left[x = \rho \sin(\alpha) \cos(\eta),\ y = \rho \sin(\alpha) \sin(\eta),\ z = h - \rho \cos(\alpha) + \frac{a}{\tan(\alpha)}\right].$$

xyz := %

Hence, the radius vector from the center of mass to a point on the cone surface is, in the conical frame,

subs(*xyz*, CartToCon(α, η) &* mat(*x*, *y*, *z*))

map(*collect*, evalm(%), [*h*, *a*], *simplify*)

r_cone := %

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} \rho \sin(\alpha)\cos(\eta) \\ \rho \sin(\alpha)\sin(\eta) \\ h - \rho\cos(\alpha) + \dfrac{a}{\tan(\alpha)} \end{bmatrix}$$

$$\begin{bmatrix} -\cos(\alpha)\, h - \dfrac{\cos(\alpha)^2\, a}{\sin(\alpha)} + \rho \\ 0 \\ \sin(\alpha)\, h + \cos(\alpha)\, a \end{bmatrix}$$

latex(mat(*xyz*), "d:/dynamics/precession/ConicalCoordinates.tex")

latex(*r_cone*, "d:/dynamics/precession/r_cone.tex")

Statement of the Torque Integral in the Conical Frame

The torque integrated over the cone surface is therefore

subs($F = K$, subsop([2, 1, 1] = '*cross*'(eval(*r_cone*), op([2, 1, 1], *Fintegral*)), *Fintegral*))

torque_cone_integral := %

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \mathrm{cross}\left(\begin{bmatrix} -\cos(\alpha)\, h - \frac{\cos(\alpha)^2\, a}{\sin(\alpha)} + \rho \\ 0 \\ \sin(\alpha)\, h + \cos(\alpha)\, a \end{bmatrix}, |P| \cos(\chi)\, \rho \sin(\alpha) \begin{bmatrix} -(A_C - 1)\, \pi_\rho \\ -(A_C - 1)\, \pi_\eta \\ -2\, A_C \cos(\chi) - (A_C - 1)\, \pi_\alpha \end{bmatrix} \right) d\rho\, d\eta$$

which can be simplified as follows.

op([2, 1, 1], *torque_cone_integral*)

$$\Big[(\sin(\alpha)\, h + \cos(\alpha)\, a) |P| \cos(\chi)\, \rho \sin(\alpha)\, (A_C - 1)\, \pi_\eta, -(\sin(\alpha)\, h + \cos(\alpha)\, a) |P| \cos(\chi)\, \rho \sin(\alpha)\, (A_C - 1)\, \pi_\rho - \left(-\cos(\alpha)\, h - \frac{\cos(\alpha)^2\, a}{\sin(\alpha)} + \rho\right) |P| \cos(\chi)\, \rho \sin(\alpha)\, (-2\, A_C \cos(\chi) - (A_C - 1)\, \pi_\alpha), -\left(-\cos(\alpha)\, h - \frac{\cos(\alpha)^2\, a}{\sin(\alpha)} + \rho\right) |P| \cos(\chi)\, \rho \sin(\alpha)\, (A_C - 1)\, \pi_\eta \Big]$$

factormat(mat(subs($A_C - 1 = -Q$, eval(%))), *collect*, [Q, π_α, π_ρ, π_η, ρ, A_C], *factor*)

$$\rho |P| \cos(\chi)\, [-(\sin(\alpha)\, h + \cos(\alpha)\, a) \sin(\alpha)\, Q\, \pi_\eta]\, [((-\rho \sin(\alpha) + \cos(\alpha)\, (\sin(\alpha)\, h + \cos(\alpha)\, a))\, \pi_\alpha + (\sin(\alpha)\, h + \cos(\alpha)\, a) \sin(\alpha)\, \pi_\rho)\, Q + 2\, \rho \sin(\alpha)\, A_C \cos(\chi) - 2 \cos(\alpha)\, (\sin(\alpha)\, h + \cos(\alpha)\, a) \cos(\chi)\, A_C]\, [(\rho \sin(\alpha) - \cos(\alpha)\, (\sin(\alpha)\, h + \cos(\alpha)\, a))\, \pi_\eta\, Q]$$

subs($\sin(\alpha)\, h + \cos(\alpha)\, a = U$, %)

factormat(%, *collect*, [Q, π_α, π_ρ, A_C], *factor*)

$\rho\,|P|\cos(\chi)$

$$\begin{bmatrix} -U\sin(\alpha)\,Q\,\pi_\eta \\ ((-\rho\sin(\alpha)+\cos(\alpha)\,U)\,\pi_\alpha + U\sin(\alpha)\,\pi_\rho)\,Q + 2\cos(\chi)\,(\rho\sin(\alpha)-\cos(\alpha)\,U)\,A_C \\ (\rho\sin(\alpha)-\cos(\alpha)\,U)\,\pi_\eta\,Q \end{bmatrix}$$

map($x \to$ algsubs($Q\,U = V$, x), map(*expand*, evalm(%)))

factormat(%, *collect*, [ρ, V], *factor*)

$\rho\,|P|\cos(\chi)$

$$\begin{bmatrix} -\sin(\alpha)\,\pi_\eta\,V \\ -\sin(\alpha)\,(\pi_\alpha\,Q - 2\,A_C\cos(\chi))\,\rho + (\sin(\alpha)\,\pi_\rho + \pi_\alpha\cos(\alpha))\,V - 2\cos(\alpha)\,U\cos(\chi)\,A_C \\ \pi_\eta\,Q\sin(\alpha)\,\rho - \pi_\eta\cos(\alpha)\,V \end{bmatrix}$$

arg := %

subslist := [$Q = 1 - A_C$, $U = \sin(\alpha)\,h + \cos(\alpha)\,a$, $V = Q\,U$]

map((x, y) $\to$ lhs(x) = subs(y, rhs(x)), *subslist*, *subslist*)

subslist := %

subslist := [$Q = 1 - A_C$, $U = \sin(\alpha)\,h + \cos(\alpha)\,a$, $V = Q\,U$]

Check:

map(*simplify*, evalm(op([2, 1, 1], *torque_cone_integral*) – subs(*subslist*, *arg*)))

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, we have

subsop([2, 1, 1] = *arg*, *torque_cone_integral*)

torque_cone_integral := %

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \rho\,|P|\cos(\chi)$$

$$\begin{bmatrix} -\sin(\alpha)\,\pi_\eta\,V \\ -\sin(\alpha)\,(\pi_\alpha\,Q - 2\,A_C\cos(\chi))\,\rho + (\sin(\alpha)\,\pi_\rho + \pi_\alpha\cos(\alpha))\,V - 2\cos(\alpha)\,U\cos(\chi)\,A_C \\ \pi_\eta\,Q\sin(\alpha)\,\rho - \pi_\eta\cos(\alpha)\,V \end{bmatrix} d\rho\,d\eta$$

where

mat(*subslist*)

$$\begin{bmatrix} Q = 1 - A_C \\ U = \sin(\alpha)\,h + \cos(\alpha)\,a \\ V = (1 - A_C)\,(\sin(\alpha)\,h + \cos(\alpha)\,a) \end{bmatrix}$$

and

$\cos(\chi) = cos_chi$

$$\cos(\chi) = -(\cos(\alpha)\cos(\eta)\,(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi)) + \cos(\alpha)\sin(\eta)\,(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi)) + \sin(\alpha)\sin(\psi)\sin(\phi))\,\pi_X - (\cos(\alpha)\cos(\eta)\,(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi)) + \cos(\alpha)\sin(\eta)\,(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi)) - \sin(\alpha)\sin(\psi)\cos(\phi))\,\pi_Y - (\cos(\alpha)\cos(\eta)\sin(\theta)\sin(\psi) + \cos(\alpha)\sin(\eta)\cos(\theta)\sin(\psi) + \sin(\alpha)\cos(\psi))\,\pi_Z$$

Partial Evaluation of the Torque Integral in the Conical Frame

torque_cone_integral

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \rho\,|P|\cos(\chi) \begin{bmatrix} -\sin(\alpha)\,\pi_\eta\,V \\ -\sin(\alpha)\,(\pi_\alpha\,Q - 2\,A_C\cos(\chi))\,\rho + (\sin(\alpha)\,\pi_\rho + \pi_\alpha\cos(\alpha))\,V - 2\cos(\alpha)\,U\cos(\chi)\,A_C \\ \pi_\eta\,Q\sin(\alpha)\,\rho - \pi_\eta\cos(\alpha)\,V \end{bmatrix} d\rho\,d\eta$$

We can do the integral over ρ right away. We note that $f = \frac{a}{\sin(\alpha)}$ and $S = \frac{b-a}{\sin(\alpha)}$. Hence,

$$\text{subs}\left(f + S = \frac{b}{\sin(\alpha)},\ f = \frac{a}{\sin(\alpha)},\ \%\right)$$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_{\frac{a}{\sin(\alpha)}}^{\frac{b}{\sin(\alpha)}} \rho \left| P \right| \cos(\chi) \begin{bmatrix} -\sin(\alpha)\, \pi_\eta\, V \\ -\sin(\alpha)\, (\pi_\alpha\, Q - 2\, A_C \cos(\chi))\, \rho + (\sin(\alpha)\, \pi_\rho + \pi_\alpha \cos(\alpha))\, V - 2 \cos(\alpha)\, U \cos(\chi)\, A_C \\ \pi_\eta\, Q \sin(\alpha)\, \rho - \pi_\eta \cos(\alpha)\, V \end{bmatrix} d\rho\, d\eta$$

Evaluating the integral in ρ, we find

$\mathrm{map}\left(int, \mathrm{map}(collect, \mathrm{evalm}(\mathrm{op}([2, 1, 1], \mathrm{subs}(\pi_\rho = Prho, torque_cone_integral))), \rho), \rho = \frac{a}{\sin(\alpha)} .. \frac{b}{\sin(\alpha)} \right)$

$\mathrm{factormat}(\mathrm{subs}(Prho = \pi_\rho, V = Q\, U, \mathrm{map}(expand, \mathrm{evalm}(\%))), collect, [\cos(\chi), \pi_\rho, \pi_\eta, \pi_\alpha, A_C, Q, U], factor)$

$\mathrm{subsop}([2, 1] = \%, \%\%\%)$

$torque_cone := \%$

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \left| P \right| \cos(\chi)\, (a - b) \begin{bmatrix} \frac{1}{2} \frac{(a+b)\, U\, Q\, \pi_\eta}{\sin(\alpha)} \\ \left[\left(\frac{(a+b) \cos(\alpha)\, U}{\sin(\alpha)^2} - \frac{2}{3} \frac{a^2 + a\, b + b^2}{\sin(\alpha)^2} \right) A_C \cos(\chi) - \frac{1}{2} \frac{(a+b)\, U\, Q\, \pi_\rho}{\sin(\alpha)} + \left(-\frac{1}{2} \frac{(a+b) \cos(\alpha)\, U}{\sin(\alpha)^2} + \frac{1}{3} \frac{a^2 + a\, b + b^2}{\sin(\alpha)^2} \right) Q\, \pi_\alpha \right] \\ \left[\left(\frac{1}{2} \frac{(a+b) \cos(\alpha)\, U}{\sin(\alpha)^2} - \frac{1}{3} \frac{a^2 + a\, b + b^2}{\sin(\alpha)^2} \right) Q\, \pi_\eta \right] \end{bmatrix} d\eta$$

Conversion of the Torque Integral to the Body Frame

Define $B_1 = \frac{(b+a)\cos(\alpha)\,U}{2\sin(\alpha)^2} - \frac{b^2+b\,a+a^2}{3\sin(\alpha)^2}$ and $B_2 = \frac{(b+a)\,U\,Q}{2\sin(\alpha)}$. Then the torque integral becomes

$$B_2 = \frac{1}{2}\frac{(a+b)\,U\,Q}{\sin(\alpha)}$$

Bsubs := [%%, %]

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} (-a+b)\cos(\chi)\,|P| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi) + B_2\,\pi_\rho + B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

Check:

map(*simplify*, evalm(subs(*Bsubs*, op([2, 1], %)) − op([2, 1], *torque_cone*)))

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

torque_cone := %%

latex(*torque_cone*, "d:/dynamics/precession/ConeTorqueIntegralCone.tex")

The pressure components in the cartesian body frame are

mat(π_x, π_y, π_z) = 'R'(ϕ, ψ, θ) &* mat(π_X, π_Y, π_Z)

$$\begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix} = \mathrm{R}(\phi, \psi, \theta)\ \&*\ \begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix}$$

Similarly, the conversion from the cartesian body frame to the conical body frame is

mat(π_ρ, π_η, π_α) = CartToCon(α, η) &* mat(π_x, π_y, π_z)

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix}\ \&*\ \begin{bmatrix} \pi_x \\ \pi_y \\ \pi_z \end{bmatrix}$$

Hence, we have

subsop([2, 2] = rhs('%%'), %)

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \left(R(\phi, \psi, \theta) \&* \begin{bmatrix} \pi_X \\ \pi_Y \\ \pi_Z \end{bmatrix} \right)$$

factormat(evalm(%), *collect*, [sin(η), cos(η)])

$$\begin{bmatrix} \pi_\rho \\ \pi_\eta \\ \pi_\alpha \end{bmatrix} =$$

$$\begin{aligned}
&[(\sin(\alpha)(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ \sin(\alpha)(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi))\pi_Y + \sin(\alpha)\cos(\theta)\sin(\psi)\pi_Z)\sin(\eta) \\
&+ (\sin(\alpha)(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ \sin(\alpha)(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\pi_Y + \sin(\alpha)\sin(\theta)\sin(\psi)\pi_Z)\cos(\eta) \\
&- \pi_X\cos(\alpha)\sin(\psi)\sin(\phi) + \pi_Y\cos(\alpha)\sin(\psi)\cos(\phi) - \pi_Z\cos(\alpha)\cos(\psi)] \\
&[((-\cos(\theta)\cos(\phi) + \sin(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ (-\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\psi)\cos(\phi))\pi_Y - \sin(\theta)\sin(\psi)\pi_Z)\sin(\eta) + (\\
&(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ (-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi))\pi_Y + \pi_Z\cos(\theta)\sin(\psi))\cos(\eta)] \\
&[(\cos(\alpha)(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ \cos(\alpha)(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi))\pi_Y + \cos(\alpha)\cos(\theta)\sin(\psi)\pi_Z)\sin(\eta) \\
&+ (\cos(\alpha)(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\pi_X \\
&+ \cos(\alpha)(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\pi_Y + \cos(\alpha)\sin(\theta)\sin(\psi)\pi_Z)\cos(\eta) \\
&+ \pi_X\sin(\alpha)\sin(\psi)\sin(\phi) - \pi_Y\sin(\alpha)\sin(\psi)\cos(\phi) + \pi_Z\sin(\alpha)\cos(\psi)]
\end{aligned}$$

Psubs := [seq(lhs(%)$_{k,1}$ = rhs(%)$_{k,1}$, $k = 1 .. 3$)]

We must transform the torque components from the conical frame to the body cartesian frame.

torque_cone

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} (b-a)\cos(\chi)\,|P| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi) + B_2\,\pi_\rho + B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

subsop([2, 1] = ConToCart(α, η) &* op([2, 1], *torque_cone*), *torque_cone*)

subs($K_\rho = K_x$, $K_\eta = K_y$, $K_\alpha = K_z$, %)

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* (b-a)\cos(\chi)\,|P| \begin{bmatrix} -B_2\,\pi_\eta \\ -2\,B_1\,A_C\cos(\chi) + B_2\,\pi_\rho + B_1\,Q\,\pi_\alpha \\ -B_1\,Q\,\pi_\eta \end{bmatrix} d\eta$$

subsop([2, 1] = factormat(evalm(op([2, 1], %)), *collect*, [π_ρ, π_η, π_α], *factor*), %)

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} |P|\cos(\chi)\,(a-b)\,[\sin(\eta)\,B_2\,\pi_\rho + \cos(\eta)\,(\sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q)\,\pi_\eta + \sin(\eta)\,B_1\,Q\,\pi_\alpha - 2\sin(\eta)\,B_1\,A_C\cos(\chi)]\,[-\cos(\eta)\,B_2\,\pi_\rho + \sin(\eta)\,(\sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q)\,\pi_\eta - \cos(\eta)\,B_1\,Q\,\pi_\alpha + 2\cos(\eta)\,B_1\,A_C\cos(\chi)]\,[(-\cos(\alpha)\,B_2 + \sin(\alpha)\,B_1\,Q)\,\pi_\eta]\,d\eta$$

foo := %

Define $C_1 = \sin(\alpha)\,B_2 + \cos(\alpha)\,B_1\,Q$ and $C_2 = -\sin(\alpha)\,Q\,B_1 + \cos(\alpha)\,B_2$. Then

Csubs := [%%, %]

invsubs(*Csubs*, *foo*)

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} |P| \cos(\chi)\,(a-b) \begin{bmatrix} \sin(\eta)\,B_2\,\pi_\rho + \cos(\eta)\,C_1\,\pi_\eta + \sin(\eta)\,B_1\,Q\,\pi_\alpha - 2\sin(\eta)\,B_1\,A_C\cos(\chi) \\ -\cos(\eta)\,B_2\,\pi_\rho + \sin(\eta)\,C_1\,\pi_\eta - \cos(\eta)\,B_1\,Q\,\pi_\alpha + 2\cos(\eta)\,B_1\,A_C\cos(\chi) \\ (-\cos(\alpha)\,B_2 + \sin(\alpha)\,B_1\,Q)\,\pi_\eta \end{bmatrix} d\eta$$

Finally, substitute for $\cos\chi$ and the pressure components and perform the integration.

```
subs(Psubs, cos(χ) = cos_chi, %)
map(int, map(collect, evalm(op([2, 1], %)), [sin(η), cos(η), |P|]), η)
foo := %
for k to 3 do foo[k, 1] := subs(η = 2 π, foo[k, 1]) − subs(η = 0, foo[k, 1]); foo[k, 1] := factor(foo[k, 1])
od
factormat(foo, collect, [π_X, π_Y, π_Z])
mat(K_x, K_y, K_z) = %
torque_xyz_cone := %
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = |P|\,\pi\,(a-b)$$

$$(-\cos(\alpha)^2\,B_2 + \sin(\alpha)^2\,B_2 + 4\sin(\alpha)\,B_1\,A_C\cos(\alpha) + C_1\sin(\alpha) + 2\sin(\alpha)\,B_1\,Q\cos(\alpha))$$
$$(\cos(\psi)\,\pi_Z - \pi_Y\sin(\psi)\cos(\phi) + \pi_X\sin(\psi)\sin(\phi))$$
$$[(\cos(\theta)\cos(\psi)\sin(\phi) + \sin(\theta)\cos(\phi))\,\pi_X$$
$$+ (-\cos(\theta)\cos(\psi)\cos(\phi) + \sin(\theta)\sin(\phi))\,\pi_Y - \pi_Z\cos(\theta)\sin(\psi)]$$
$$[(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y$$
$$+ \sin(\theta)\sin(\psi)\,\pi_Z]$$
$$[0]$$

```
latex(torque_xyz_cone, "d:/dynamics/precession/ConeTorqueBody.tex")
```

We can simplify C_1 and C_2 somewhat:

$\text{subs}(\sin(\alpha)^2 + \cos(\alpha)^2 = 1, \text{collect}(\text{subs}(Bsubs, Csubs_1), [Q, U], factor))$

$\text{subs}(subslist_1, \text{collect}(\%, [\sin(\alpha)]))$

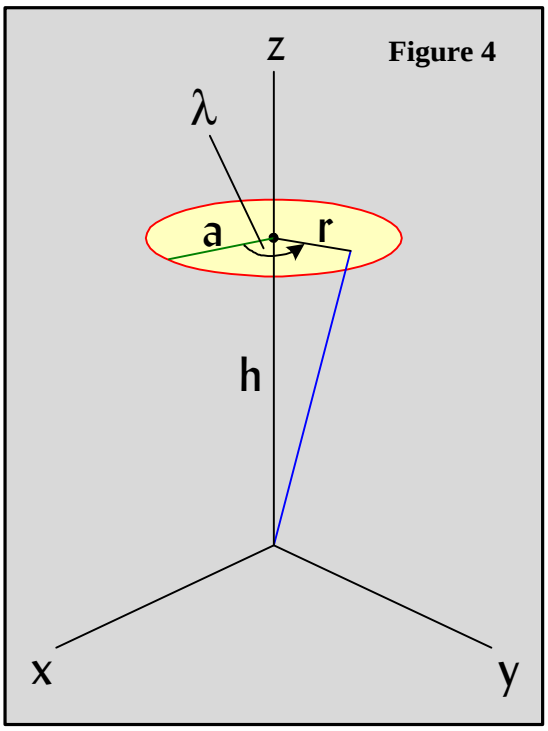
$$C_1 = \frac{\left(\frac{1}{2}(a+b)\,U - \frac{1}{3}\cos(\alpha)\,(a^2 + a\,b + b^2)\right)(1 - A_C)}{\sin(\alpha)^2}$$

$\text{subs}(subslist_1, \text{factor}(\text{subs}(Bsubs, Csubs_2)))$

$$C_2 = \frac{1}{3}\,\frac{(1 - A_C)\,(a^2 + a\,b + b^2)}{\sin(\alpha)}$$

$Csubs := [\%\%, \%]$

Torque Due to Radiation Pressure on the Flattop Surface



$\text{subs}(Csubs, Bsubs, subslist, A_C = A_T, a = 0, b = a, torque_xyz_cone)\Big|_{\alpha = \frac{\pi}{2}}$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = -|P|\,\pi\,a^2\,h\,(1 - A_T)\,(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))$$

$$[(\cos(\theta)\cos(\psi)\sin(\phi)+\sin(\theta)\cos(\phi))\,\pi_X+(-\cos(\theta)\cos(\psi)\cos(\phi)+\sin(\theta)\sin(\phi))\,\pi_Y - \pi_Z\cos(\theta)\sin(\psi)]$$

$$[(\cos(\theta)\cos(\phi)-\sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X+(\cos(\theta)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y + \sin(\theta)\sin(\psi)\,\pi_Z]$$

$$[0]$$

torque_xyz_top := %

latex(*torque_xyz_top*, "d:/dynamics/precession/FlatTopTorqueBody.tex")

The Equations of Motion

The equations of motion are

mat(*FirstOrderODEsK*)

$$\left[\frac{\partial}{\partial t}\phi=\Omega_\phi\right]$$

$$\left[\frac{\partial}{\partial t}\psi=\Omega_\psi\right]$$

$$\left[\frac{\partial}{\partial t}\theta=\Omega_\theta\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right)=((1-\beta)\,\Omega_\theta-\cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi+\frac{\cos(\theta)\,K_y+K_x\sin(\theta)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\psi=\left((-1+\beta)\,\Omega_\phi\,\Omega_\theta+\Omega_\phi^{\,2}\cos(\psi)\,\beta\right)\sin(\psi)+\frac{-K_y\sin(\theta)+\cos(\theta)\,K_x}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right)=((\cos(\psi)^2\,\beta+1)\,\Omega_\phi+(-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi + \frac{-\frac{\sin(\psi)\,K_z}{-1+\beta}-(\cos(\theta)\,K_y+K_x\sin(\theta))\cos(\psi)}{I_{xy}}\right]$$

Torque Combinations

Assemble the torque combinations.

Kcone := convert(evalm(rhs(*torque_xyz_cone*)), *vector*)

Ktop := convert(evalm(rhs(*torque_xyz_top*)), *vector*)

$$(Kcone_1+Ktop_1)\sin(\theta)+(Kcone_2+Ktop_2)\cos(\theta)$$

$$\text{applyrule}(\sin(\theta)^2 + \cos(\theta)^2 = 1, \text{factor}(\%))$$

$$\text{map}(collect, \%, [B_1, B_2, C_1], factor)$$

$$torque_\phi := \%$$

$$|P|\,\pi\,(\pi_X \cos(\phi) + \pi_Y \sin(\phi))\,(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,(2\sin(\alpha)\cos(\alpha)\,(a-b)\,(Q + 2\,A_C)\,B_1 + (\sin(\alpha) - \cos(\alpha))\,(\sin(\alpha) + \cos(\alpha))\,(a-b)\,B_2 + \sin(\alpha)\,(a-b)\,C_1 + a^2\,h\,(-1 + A_T))$$

and

$$(Kcone_1 + Ktop_1)\cos(\theta) - (Kcone_2 + Ktop_2)\sin(\theta)$$

$$\text{applyrule}(\sin(\theta)^2 + \cos(\theta)^2 = 1, \text{factor}(\%))$$

$$\text{map}(collect, \%, [B_1, B_2, C_1], factor)$$

$$torque_\psi := \%$$

$$-|P|\,\pi\,(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,(\sin(\psi)\,\pi_Z + \cos(\psi)\,\pi_Y \cos(\phi) - \sin(\phi)\cos(\psi)\,\pi_X)\,(2\sin(\alpha)\cos(\alpha)\,(a-b)\,(Q + 2\,A_C)\,B_1 + (\sin(\alpha) - \cos(\alpha))\,(\sin(\alpha) + \cos(\alpha))\,(a-b)\,B_2 + \sin(\alpha)\,(a-b)\,C_1 + a^2\,h\,(-1 + A_T))$$

where

Csubs

$$\left[C_1 = \frac{\left(\frac{1}{2}(a+b)\,U - \frac{1}{3}\cos(\alpha)\,(a^2 + a\,b + b^2)\right)(1 - A_C)}{\sin(\alpha)^2},\ C_2 = \frac{1}{3}\,\frac{(1 - A_C)\,(a^2 + a\,b + b^2)}{\sin(\alpha)}\right]$$

Bsubs

$$\left[B_1 = \frac{1}{2}\frac{(a+b)\cos(\alpha)\,U}{\sin(\alpha)^2} - \frac{1}{3}\frac{a^2+a\,b+b^2}{\sin(\alpha)^2},\ B_2 = \frac{1}{2}\frac{(a+b)\,U\,Q}{\sin(\alpha)}\right]$$

subslist

$$[Q = 1 - A_C,\ U = \sin(\alpha)\,h + \cos(\alpha)\,a,\ V = (1 - A_C)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)]$$

Simplification of the Torque Combinations

Let's check the commonality between the two torque combinations.

$\mathrm{mat}(K_x \sin(\theta) + K_y \cos(\theta), K_x \cos(\theta) - K_y \sin(\theta)) =$
$\mathrm{factormat}(\mathrm{subs}(subslist_1, \mathrm{mat}(torque_\phi, torque_\psi)), collect, [B_1, B_2, C_1], factor)$

$$\begin{bmatrix} \cos(\theta)\,K_y + K_x \sin(\theta) \\ -K_y \sin(\theta) + \cos(\theta)\,K_x \end{bmatrix} = |P|\,\pi\,(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,(2\sin(\alpha)\cos(\alpha)\,(a-b)\,(1+A_C)\,B_1 + (\sin(\alpha)-\cos(\alpha))\,(\sin(\alpha)+\cos(\alpha))\,(a-b)\,B_2 + \sin(\alpha)\,(a-b)\,C_1 + a^2\,h\,(-1+A_T))\begin{bmatrix} \pi_X \cos(\phi) + \pi_Y \sin(\phi) \\ -\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X \end{bmatrix}$$

torque_terms := %

$\mathrm{select}(has, \mathrm{rhs}(\%), a)$

$$2\sin(\alpha)\cos(\alpha)\,(a-b)\,(1+A_C)\,B_1 + (\sin(\alpha)-\cos(\alpha))\,(\sin(\alpha)+\cos(\alpha))\,(a-b)\,B_2 + \sin(\alpha)\,(a-b)\,C_1 + a^2\,h\,(-1+A_T)$$

$L := \mathrm{location}(\%\%, \%)$

$$L := [2, 4]$$

$\mathrm{subs}(Csubs, Bsubs, subslist_1, \%\%)$

$$2\sin(\alpha)\cos(\alpha)\,(a-b)\,(1+A_C)\left(\frac{1}{2}\frac{(a+b)\cos(\alpha)\,U}{\sin(\alpha)^2} - \frac{1}{3}\frac{a^2+a\,b+b^2}{\sin(\alpha)^2}\right) + \frac{1}{2}\frac{(\sin(\alpha)-\cos(\alpha))\,(\sin(\alpha)+\cos(\alpha))\,(a-b)\,(a+b)\,U\,(1-A_C)}{\sin(\alpha)} + \frac{(a-b)\left(\frac{1}{2}(a+b)\,U - \frac{1}{3}\cos(\alpha)\,(a^2+a\,b+b^2)\right)(1-A_C)}{\sin(\alpha)} + a^2\,h\,(-1+A_T)$$

bar := %

latex(%, "d:/dynamics/precession/torque_term_to_simplify.tex")

Collect(%, [$b-a$, $b+a$, $b^2+b\,a+a^2$, U, A_C], *simplify*, *loc*)

$$\left(\frac{(-a+2\cos(\alpha)^2\,a+b-2\cos(\alpha)^2\,b)\,A_C}{\sin(\alpha)}+\frac{a-b}{\sin(\alpha)}\right)U\,(a+b)+\left(-\frac{1}{3}\frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)}-\frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)}\right)(a^2+a\,b+b^2)+a^2\,h\,(-1+A_T)$$

location(%, remove(*has*, select(*has*, %, U), $b-a$))

$$[1]$$

subsop(% = map(*factor*, op(%, %%)), %%)

$$\frac{(a-b)\,(-A_C+1+2\,A_C\cos(\alpha)^2)\,U\,(a+b)}{\sin(\alpha)}+\left(-\frac{1}{3}\frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)}-\frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)}\right)(a^2+a\,b+b^2)+a^2\,h\,(-1+A_T)$$

latex(%, "d:/dynamics/precession/torque_term_simplified.tex")

Check:

simplify(% − *bar*)

$$0$$

subs(*subslist*, subsop(L = %%, *torque_terms*))

$$\begin{bmatrix}\cos(\theta)\,K_y+K_x\sin(\theta)\\ -K_y\sin(\theta)+\cos(\theta)\,K_x\end{bmatrix}=|P|\,\pi\,(\cos(\psi)\,\pi_Z-\pi_Y\sin(\psi)\cos(\phi)+\pi_X\sin(\psi)\sin(\phi))\left(\frac{(a-b)\,(-A_C+1+2\,A_C\cos(\alpha)^2)\,(\sin(\alpha)\,h+\cos(\alpha)\,a)\,(a+b)}{\sin(\alpha)}+\left(-\frac{1}{3}\frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)}-\frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)}\right)(a^2+a\,b+b^2)+a^2\,h\,(-1+A_T)\right)\begin{bmatrix}\pi_X\cos(\phi)+\pi_Y\sin(\phi)\\ -\sin(\psi)\,\pi_Z-\cos(\psi)\,\pi_Y\cos(\phi)+\sin(\phi)\cos(\psi)\,\pi_X\end{bmatrix}$$

torque_terms := %

This looks good. Now define

select(*has*, rhs(*torque_terms*), ψ)

$$(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))$$

$$\begin{bmatrix} \pi_X \cos(\phi) + \pi_Y \sin(\phi) \\ -\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X \end{bmatrix}$$

select(*has*, %, $\sin(\psi)\,\pi_Z$)

$$\begin{bmatrix} \pi_X \cos(\phi) + \pi_Y \sin(\phi) \\ -\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X \end{bmatrix}$$

gsubs := [$g_0(\phi, \psi)$ = remove(*has*, %%, $\sin(\psi)\,\pi_Z$), $g_1(\phi, \psi) = \%_{1,1}$, $g_2(\phi, \psi) = \%_{2,1}$, $G(a, b, h, A_C, A_T, \alpha)$ = select(*has*, remove(*has*, rhs(*torque_terms*), ψ), { a, P, π })]

Then we can write

invsubs(*gsubs*, remove(*has*, rhs(*torque_terms*), ψ))

$$G(a, b, h, A_C, A_T, \alpha)$$

invsubs(*gsubs*, select(*has*, rhs(*torque_terms*), ψ))

$$g_0(\phi, \psi)\begin{bmatrix} g_1(\phi, \psi) \\ g_2(\phi, \psi) \end{bmatrix}$$

lhs(*torque_terms*) = % %%

$$\begin{bmatrix} \cos(\theta)\,K_y + K_x \sin(\theta) \\ -K_y \sin(\theta) + \cos(\theta)\,K_x \end{bmatrix} = g_0(\phi, \psi)\begin{bmatrix} g_1(\phi, \psi) \\ g_2(\phi, \psi) \end{bmatrix} G(a, b, h, A_C, A_T, \alpha)$$

torque_terms_subs := %

where

mat(*gsubs*)

$$[g_0(\phi, \psi) = \cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi)]$$

$$[g_1(\phi, \psi) = \pi_X \cos(\phi) + \pi_Y \sin(\phi)]$$

$$[g_2(\phi, \psi) = -\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X]$$

$$\left[G(a, b, h, A_C, A_T, \alpha) = |P|\,\pi \left(\frac{(a-b)\,(-A_C + 1 + 2\,A_C \cos(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(a+b)}{\sin(\alpha)} \right.\right.$$

$$+\left(-\frac{1}{3}\frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)}-\frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)}\right)(a^2+a\,b+b^2)+a^2\,h\,(-1+A_T)\Bigg)\Bigg]$$

Equations of Motion

subs(seq(lhs($torque_terms_subs$)$_{k,1}$ = evalm(rhs($torque_terms_subs$))$_{k,1}$, $k = 1 .. 2$), $K_z = 0$, $FirstOrderODEsK$)

mat(%)

$$\left[\frac{\partial}{\partial t}\phi=\Omega_\phi\right]$$

$$\left[\frac{\partial}{\partial t}\psi=\Omega_\psi\right]$$

$$\left[\frac{\partial}{\partial t}\theta=\Omega_\theta\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right)=((1-\beta)\,\Omega_\theta-\cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi+\frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)}{I_{xy}}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\psi=\left((-1+\beta)\,\Omega_\phi\,\Omega_\theta+\Omega_\phi^{\,2}\cos(\psi)\,\beta\right)\sin(\psi)+\frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_2(\phi,\psi)}{I_{xy}}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right)=((\cos(\psi)^2\,\beta+1)\,\Omega_\phi+(-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi-\frac{g_0(\phi,\psi)\,\mathrm{G}(a,b,h,A_C,A_T,\alpha)\,g_1(\phi,\psi)\cos(\psi)}{I_{xy}}\right]$$

$FirstOrderODEs$:= %

where

mat($gsubs$)

$$[g_0(\phi,\psi)=\cos(\psi)\,\pi_Z-\pi_Y\sin(\psi)\cos(\phi)+\pi_X\sin(\psi)\sin(\phi)]$$

$$[g_1(\phi,\psi)=\pi_X\cos(\phi)+\pi_Y\sin(\phi)]$$

$$[g_2(\phi,\psi)=-\sin(\psi)\,\pi_Z-\cos(\psi)\,\pi_Y\cos(\phi)+\sin(\phi)\cos(\psi)\,\pi_X]$$

$$\left[G(a, b, h, A_C, A_T, \alpha) = |P|\, \pi \left(\frac{(a-b)\,(-A_C + 1 + 2\,A_C \cos(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(a+b)}{\sin(\alpha)} + \left(-\frac{1}{3} \frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)} - \frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)} \right) (a^2 + a\,b + b^2) + a^2\,h\,(-1 + A_T) \right) \right]$$

```
latex(FirstOrderODEs, "d:/dynamics/precession/FirstOrderODEs.tex");
latex(mat(gsubs), "d:/dynamics/precession/gsubs.tex")
```

Fast Spin Approximation

The equations of motion are

$$eqs := \mathrm{subs}(\mathrm{op}(\mathrm{select}(has, gsubs, \psi)), G(a, b, h, A_C, A_T, \alpha) = G, [\mathrm{seq}(FirstOrderODEs_{k,1}, k = 4 .. 6)])$$

$$\mathrm{mat}(\%)$$

$$\left[\sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\phi \right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G\,(\pi_X \cos(\phi) + \pi_Y \sin(\phi))}{I_{xy}} \right]$$

$$\left[\frac{\partial}{\partial t} \Omega_\psi = \left((-1+\beta)\,\Omega_\phi\,\Omega_\theta + \Omega_\phi^{\,2} \cos(\psi)\,\beta \right) \sin(\psi) + ((\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G\,(-\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X)) \Big/ I_{xy} \right]$$

$$\left[\sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\theta \right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi - \frac{(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G\,(\pi_X \cos(\phi) + \pi_Y \sin(\phi))\cos(\psi)}{I_{xy}} \right]$$

For purposes of substitution, which we'll use in a moment, divide by $\sin(\psi)$ where appropriate.

$$\mathrm{select}\left(has, eqs, \frac{\partial}{\partial t} \Omega_\phi \right)$$

$L := \mathrm{location}(eqs, \mathrm{op}(\%))$

$\mathrm{subsop}\left(\% = \frac{\mathrm{op}(\%\%)}{\sin(\psi)}, eqs\right)$

$\mathrm{select}\left(has, \%, \frac{\partial}{\partial t}\Omega_\theta\right)$

$L := \mathrm{location}(\%\%, \mathrm{op}(\%))$

$\mathrm{subsop}\left(\% = \frac{\mathrm{op}(\%\%)}{\sin(\psi)}, \%\%\%\right)$

$odesubs := \%$

$\mathrm{mat}(\%)$

$$\left[\frac{\partial}{\partial t}\Omega_\phi = \left(((1-\beta)\,\Omega_\theta - \cos(\psi)\,(1+\beta)\,\Omega_\phi)\,\Omega_\psi + \frac{(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G\,(\pi_X\cos(\phi) + \pi_Y \sin(\phi))}{I_{xy}}\right) / \sin(\psi)\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\psi = \left((-1+\beta)\,\Omega_\phi\,\Omega_\theta + \Omega_\phi^{\,2}\cos(\psi)\,\beta\right)\sin(\psi) + (\right.$$
$$(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G$$
$$\left.(-\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X)\right) \Big/ I_{xy}\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_\theta = \left(((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi - \frac{(\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G\,(\pi_X\cos(\phi) + \pi_Y \sin(\phi))\cos(\psi)}{I_{xy}}\right) / \sin(\psi)\right]$$

Differentiate the equations of motion.

$\mathrm{map}(diff, eqs, t)$

$eqs2 := \%$

$\mathrm{mat}(\%)$

$$\left[\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\Omega_\phi\right) + \sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\phi\right) =\right.$$

$$\left((1-\beta)\left(\frac{\partial}{\partial t}\Omega_\theta\right)+\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)(1+\beta)\,\Omega_\phi-\cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\Omega_\phi\right)\right)\Omega_\psi$$

$$+((1-\beta)\,\Omega_\theta-\cos(\psi)\,(1+\beta)\,\Omega_\phi)\left(\frac{\partial}{\partial t}\Omega_\psi\right)+\left(\left(-\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Z\right.\right.$$

$$-\pi_Y\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\cos(\phi)+\pi_Y\sin(\psi)\sin(\phi)\left(\frac{\partial}{\partial t}\phi\right)+\pi_X\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\sin(\phi)$$

$$\left.\left.+\pi_X\sin(\psi)\cos(\phi)\left(\frac{\partial}{\partial t}\phi\right)\right)G\,(\pi_X\cos(\phi)+\pi_Y\sin(\phi))\right)\Big/\,I_{xy}+\Bigg($$

$$(\cos(\psi)\,\pi_Z-\pi_Y\sin(\psi)\cos(\phi)+\pi_X\sin(\psi)\sin(\phi))\,G\left(-\pi_X\sin(\phi)\left(\frac{\partial}{\partial t}\phi\right)+\pi_Y\cos(\phi)\left(\frac{\partial}{\partial t}\phi\right)\right)$$

$$\Bigg)\Big/\,I_{xy}\Bigg]$$

$$\Bigg[\frac{\partial^2}{\partial t^2}\Omega_\psi=\Bigg($$

$$(-1+\beta)\left(\frac{\partial}{\partial t}\Omega_\phi\right)\Omega_\theta+(-1+\beta)\,\Omega_\phi\left(\frac{\partial}{\partial t}\Omega_\theta\right)+2\,\Omega_\phi\cos(\psi)\,\beta\left(\frac{\partial}{\partial t}\Omega_\phi\right)-\Omega_\phi{}^2\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\beta$$

$$\Bigg)\sin(\psi)+\left((-1+\beta)\,\Omega_\phi\,\Omega_\theta+\Omega_\phi{}^2\cos(\psi)\,\beta\right)\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\left(\left(-\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Z\right.\right.$$

$$-\pi_Y\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\cos(\phi)+\pi_Y\sin(\psi)\sin(\phi)\left(\frac{\partial}{\partial t}\phi\right)+\pi_X\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\sin(\phi)$$

$$\left.\left.+\pi_X\sin(\psi)\cos(\phi)\left(\frac{\partial}{\partial t}\phi\right)\right)G\,(-\sin(\psi)\,\pi_Z-\cos(\psi)\,\pi_Y\cos(\phi)+\sin(\phi)\cos(\psi)\,\pi_X)\right)\Big/\,I_{xy}$$

$$+\Bigg((\cos(\psi)\,\pi_Z-\pi_Y\sin(\psi)\cos(\phi)+\pi_X\sin(\psi)\sin(\phi))\,G\left(-\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Z\right.$$

$$+\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Y\cos(\phi)+\cos(\psi)\,\pi_Y\sin(\phi)\left(\frac{\partial}{\partial t}\phi\right)+\cos(\phi)\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)\,\pi_X$$

$$\left.-\sin(\phi)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_X\right)\Bigg)\Big/\,I_{xy}\Bigg]$$

$$\Bigg[\cos(\psi)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\Omega_\theta\right)+\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\theta\right)=\Bigg(-2\cos(\psi)\,\beta\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\Omega_\phi$$

$$+(\cos(\psi)^2\,\beta+1)\left(\frac{\partial}{\partial t}\Omega_\phi\right)-(-1+\beta)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\Omega_\theta+(-1+\beta)\cos(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right)\Bigg)\Omega_\psi$$

$$+((\cos(\psi)^2\,\beta+1)\,\Omega_\phi+(-1+\beta)\cos(\psi)\,\Omega_\theta)\left(\frac{\partial}{\partial t}\Omega_\psi\right)-\left(\left(-\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\pi_Z\right.\right.$$

$$- \pi_Y \cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \cos(\phi) + \pi_Y \sin(\psi) \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) + \pi_X \cos(\psi) \left(\frac{\partial}{\partial t} \psi\right) \sin(\phi)$$

$$+ \pi_X \sin(\psi) \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right)\Bigg) G \left(\pi_X \cos(\phi) + \pi_Y \sin(\phi)\right) \cos(\psi)\Bigg) \Big/ I_{xy} - \Bigg($$

$$\left(\cos(\psi) \pi_Z - \pi_Y \sin(\psi) \cos(\phi) + \pi_X \sin(\psi) \sin(\phi)\right) G \left(-\pi_X \sin(\phi) \left(\frac{\partial}{\partial t} \phi\right) + \pi_Y \cos(\phi) \left(\frac{\partial}{\partial t} \phi\right)\right)$$

$$\cos(\psi)\Bigg) \Big/ I_{xy} +$$

$$\frac{\left(\cos(\psi) \pi_Z - \pi_Y \sin(\psi) \cos(\phi) + \pi_X \sin(\psi) \sin(\phi)\right) G \left(\pi_X \cos(\phi) + \pi_Y \sin(\phi)\right) \sin(\psi) \left(\frac{\partial}{\partial t} \psi\right)}{I_{xy}}$$

$$\Bigg]$$

Substitute for $\frac{\partial}{\partial t} \Omega_\theta$, $\frac{\partial}{\partial t} \Omega_\phi$, and $\frac{\partial}{\partial t} \Omega_\psi$ using the original equations of motion. Then, assume Ω_θ is large and the pressure terms (i.e., $G(a, b, h, A_C, A_T, \alpha)$) are small. We find the resulting second-order differential equations,

```
subsODEs := proc(ode, odesubs)
local q, Lq, u, Lu, eqn;
    eqn := copy(ode);
    if not (isdiff(eqn) and 1 < difforder(eqn)) then for q in eqn do
            Lq := location(eqn, q);
            if isdiff(q) and difforder(q) = 1 then q := subs(odesubs, q)
            elif 1 < nops(q) and not isdiff(u) then q := procname(q, odesubs)
            fi;
            eqn := subsop(Lq = q, eqn)
        od
    fi;
    eval(eqn)
end
```

Check:

```
for ii to 3 do factor(subs(odesubs, rhs(eqs2_ii)) - subsODEs(rhs(eqs2_ii), odesubs)) od
```

$$0$$

$$0$$

$$0$$

```
subsODEs(eqs2, odesubs)
```

$$\mathrm{subs}\left(\frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\phi = \Omega_\phi, \%\right)$$

$$\mathrm{convert}(\mathrm{expansion}(\%, [\Omega_\psi, \Omega_\phi, G], 1), diff)$$

$$\mathrm{Collect}(\%, [G], factor)$$

$$eqs2 := \mathrm{invsubs}([\mathrm{op}(gsubs), \mathrm{lhs}(\mathrm{op}(\mathrm{select}(has, gsubs, G))) = G], \%)$$

$$\mathrm{mat}(\%)$$

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\phi\right) = -\frac{(-1+\beta)\,\Omega_\theta\, g_0(\phi,\psi)\, g_2(\phi,\psi)\, \mathrm{G}(a, b, h, A_C, A_T, \alpha)}{I_{xy}} - (-1+\beta)^2\,\Omega_\theta{}^2 \sin(\psi)\,\Omega_\phi\right]$$

$$\left[\frac{\partial^2}{\partial t^2}\Omega_\psi = \frac{(-1+\beta)\, g_0(\phi,\psi)\, \mathrm{G}(a, b, h, A_C, A_T, \alpha)\, g_1(\phi,\psi)\,\Omega_\theta}{I_{xy}} - (-1+\beta)^2\,\Omega_\theta{}^2\,\Omega_\psi\right]$$

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\theta\right) = (-1+\beta)^2 \cos(\psi)\,\Omega_\theta{}^2 \sin(\psi)\,\Omega_\phi + \frac{(-1+\beta)\cos(\psi)\,\Omega_\theta\, g_0(\phi,\psi)\, \mathrm{G}(a, b, h, A_C, A_T, \alpha)\, g_2(\phi,\psi)}{I_{xy}}\right]$$

Since Ω_θ is large, we may also assume its rate of change is small, hence $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$. The first consequence of this is that the angular velocity of the inclination, Ω_ψ, exhibits simple harmonic motion: the solution of

$$\left(\frac{\partial^2}{\partial t^2}\Omega_\psi\right) + (1-\beta)^2\,\Omega_\theta{}^2\,\Omega_\psi + \frac{(1-\beta)\, g_0(\phi,\psi)\, \mathrm{G}(a, b, h, A_C, A_T, \alpha)\, g_1(\phi,\psi)\,\Omega_\theta}{I_{xy}} = 0$$

is just

$$\mathrm{collect}(\mathrm{subs}(\omega = \Omega_\theta, g_0 = g_0(\phi,\psi), g_1 = g_1(\phi,\psi),$$
$$\mathrm{dsolve}(\mathrm{subs}(\Omega_\theta = \omega, g_1(\phi,\psi) = g_1, g_0(\phi,\psi) = g_0, \%), \Omega_\psi)), [_C1, _C2], factor)$$

$$\Omega_\psi = \frac{g_0(\phi,\psi)\, \mathrm{G}(a, b, h, A_C, A_T, \alpha)\, g_1(\phi,\psi)}{\Omega_\theta\,(-1+\beta)\, I_{xy}} + _C1 \cos((-1+\beta)\,\Omega_\theta\, t) + _C2 \sin((-1+\beta)\,\Omega_\theta\, t)$$

which is simple harmonic motion with a small linear drift in $\psi(t)$. The second consequence stems from either the first or the third equation (both say the same thing). Setting $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$ places a constraint on the value of the longitudinal motion — i.e., the precession rate. We then have

subs(select(*has*, *gsubs*, *g*), isolate(rhs(*eqs2*$_3$), Ω_ϕ))

precession_rate := %

$$\Omega_\phi = -\left((\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G(a, b, h, A_C, A_T, \alpha)\right.$$
$$\left.(-\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X)\right) \Big/ \left((-1+\beta)\,\Omega_\theta\, I_{xy} \sin(\psi)\right)$$

To make this expression a little bit clearer, further assume that the pressure is mainly along the Z axis — that is, that π_X and π_Y are small. Then this expression becomes

$$\text{Collect}\left(\text{expansion}(\%, [\pi_X, \pi_Y], 1), \left[G(a, b, h, A_C, A_T, \alpha), \pi, |P|, \frac{1}{\Omega_\theta}, \frac{1}{I_{xy}}, -1+\beta, \pi_Z\right], \textit{factor}\right)$$

$$\Omega_\phi = \left(\left(\cos(\psi)\,\pi_Z^2 - \frac{(\cos(\psi) - \sin(\psi))\,(\cos(\psi) + \sin(\psi))\,(\pi_X \sin(\phi) - \pi_Y \cos(\phi))\,\pi_Z}{\sin(\psi)}\right)\right.$$
$$\left. G(a, b, h, A_C, A_T, \alpha)\right) \Big/ \left((-1+\beta)\, I_{xy}\, \Omega_\theta\right)$$

Thus, we have a weak phase dependence on the non-axial pressure components. For the purely axial case $\pi_Z = 1$, we have

subs($\pi_Z = 1$, $\pi_X = 0$, $\pi_Y = 0$, %)

$$\Omega_\phi = \frac{\cos(\psi)\,G(a, b, h, A_C, A_T, \alpha)}{(-1+\beta)\, I_{xy}\, \Omega_\theta}$$

Finally, the precession for a flat disk of uniform albedo, $\alpha = \frac{\pi}{2}$, $A_C = A$, and $A_T = A$, is

$$\text{factor}\left(\text{eval}\left(\text{subs}\left(\textit{gsubs}, \alpha = \frac{\pi}{2}, A_C = A, A_T = A, \%\right)\right)\right)$$

$$\Omega_\phi = \frac{\cos(\psi)\,|P|\,\pi\, h\, b^2\,(-1+A)}{(-1+\beta)\,\Omega_\theta\, I_{xy}}$$

Precession Null

Numerical

We can determine the approximate angles α at which the precession is zero. From

precession_rate

$$\Omega_\phi = -\left((\cos(\psi)\,\pi_Z - \pi_Y \sin(\psi)\cos(\phi) + \pi_X \sin(\psi)\sin(\phi))\,G(a, b, h, A_C, A_T, \alpha)\right.$$
$$\left.(-\sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi) + \sin(\phi)\cos(\psi)\,\pi_X)\right) \Big/ \left((-1+\beta)\,\Omega_\theta\, I_{xy} \sin(\psi)\right)$$

we see that for $\Omega_\phi = 0$ we must have $G(a, b, h, A_C, A_T, \alpha) = 0$. Hence, we require

$$\frac{\mathrm{rhs}(\mathrm{op}(\mathrm{select}(has, gsubs, G)))}{\pi\,|P|} = 0$$

$$\frac{(a-b)\,(-A_C + 1 + 2\,A_C \cos(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(a+b)}{\sin(\alpha)}$$
$$+\left(-\frac{1}{3}\,\frac{\cos(\alpha)\,(a-b)\,A_C}{\sin(\alpha)} - \frac{\cos(\alpha)\,(a-b)}{\sin(\alpha)}\right)(a^2 + a\,b + b^2) + a^2\,h\,(-1 + A_T) = 0$$

latex(%, "d:/dynamics/precession/PrecessionNullEq.tex")

Plots

Analytic

save "d:/dynamics/precession/precession.m"

restart

alias($I = I$)

alias($\psi = \psi(t)$, $\theta = \theta(t)$, $\phi = \phi(t)$, $\Omega_x = \Omega_x(t)$, $\Omega_y = \Omega_y(t)$, $\Omega_z = \Omega_z(t)$, $\Omega_\phi = \Omega_\phi(t)$, $\Omega_\psi = \Omega_\psi(t)$, $\Omega_\theta = \Omega_\theta(t)$)

read "d:/dynamics/precession/precession.m"

Id, fn, det, size, rows, cols, transpose, inverse, augment, stack, extend, grad, curl, div, laplacian, angle, intparts, ψ, θ, ϕ, Ω_x, Ω_y, Ω_z, Ω_ϕ, Ω_ψ, Ω_θ

?