An Automated Linear Least Squares Solution Generator

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1. Introduction

In day-to-day work, one often has need to perform a quick linear least squares calculation for some model which approximates a physical process. Since the model is likely to be something other than simple linear regression, one winds up (often re-) deriving the appropriate normal equations and solving for the model parameters. This is wasted effort, thanks to the availability of computer algebra systems, in which it is easy to automate this process. In the section which follows, I illustrate the basic "quick and dirty" linear least squares process by means of a simple example. In section 3, I present a Maple procedure that performs the algebra autmatically for any model that is linear in the parameters, and in section 4 are a few examples of its usage.

2. A Linear Least Squares Example

Start with a simple second degree polynomial as our model: $model := A + B\,t_i + C\,t_i^2$. Then the χ^2 statistic is $\chi^2 = \sum_{i=1}^{N} (y_i - model)^2$.

$$\chi^2 = \sum_{i=1}^{N} \left(y_i - A - B\,t_i - C\,t_i^2 \right)^2$$

where the sum is over N available data points, and the y_i are the data values, measured at times t_i. In keeping with the "quick and dirty" premise, we do not take into account *a priori* any weighting of the data. It is a simple matter to add weight factors into the solutions afterwards; including them now would serve only to clutter up the notation.

Next, we calculate the partial derivatives of χ^2 with respect to the model parameters, setting these derivatives to zero and thus forming the normal equations.

$$\frac{\partial}{\partial A}\,\mathrm{rhs}(\%) = 0$$

$$\frac{\partial}{\partial B}\,\mathrm{rhs}(\%\%) = 0$$

$$\frac{\partial}{\partial C}\,\mathrm{rhs}(\%\%\%) = 0$$

$$\sum_{i=1}^{N}\left(-2\,y_i + 2\,A + 2\,B\,t_i + 2\,C\,t_i^{\,2}\right) = 0$$

$$\sum_{i=1}^{N}\left(-2\left(y_i - A - B\,t_i - C\,t_i^{\,2}\right)t_i\right) = 0$$

$$\sum_{i=1}^{N}\left(-2\left(y_i - A - B\,t_i - C\,t_i^{\,2}\right)t_i^{\,2}\right) = 0$$

NormalEqs := { %, %%, %%% }

Finally, we solve the normal equations for the parameter values. These values are the linear least squares solution.

sols := sortsols(solve(value(*NormalEqs*), { *A*, *B*, *C* }), [*A*, *B*, *C*])

for *p* **in** *sols* **do** print(*p*) **od**

$$A = \left(\left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2 - \left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right) + \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i y_i\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right) - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right) - \left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i y_i\right) + \left(\sum_{i=1}^{N} t_i^{\,2}\right)^2\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right)\right) \Big/ \left(-2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right) + N\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2 + \left(\sum_{i=1}^{N} t_i^{\,2}\right)^3 + \left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i\right)^2 - \left(\sum_{i=1}^{N} t_i^{\,4}\right) N \left(\sum_{i=1}^{N} t_i^{\,2}\right)\right)$$

$$B = \left(\left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i\right) - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right) - \left(\sum_{i=1}^{N} t_i y_i\right) N \left(\sum_{i=1}^{N} t_i^{\,4}\right) - \left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right) + N\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right)\right)$$

$$+\left(\sum_{i=1}^{N} t_i^{\,2}\right)^2\left(\sum_{i=1}^{N} t_i y_i\right)\Bigg)\Bigg/\left(-2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)+N\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2\right.$$

$$\left.+\left(\sum_{i=1}^{N} t_i^{\,2}\right)^3+\left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i\right)^2-\left(\sum_{i=1}^{N} t_i^{\,4}\right)N\left(\sum_{i=1}^{N} t_i^{\,2}\right)\right)$$

$$C=\left(-\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i y_i\right)+N\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i y_i\right)\right.$$

$$-\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i\right)+\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right)\left(\sum_{i=1}^{N} t_i\right)^2-\left(\sum_{i=1}^{N} t_i^{\,2} y_i\right)N\left(\sum_{i=1}^{N} t_i^{\,2}\right)$$

$$+\left(\sum_{i=1}^{N} t_i^{\,2}\right)^2\left(\sum_{i=1}^{N} y_i\right)\Bigg)\Bigg/\left(-2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)+N\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2\right.$$

$$\left.+\left(\sum_{i=1}^{N} t_i^{\,2}\right)^3+\left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i\right)^2-\left(\sum_{i=1}^{N} t_i^{\,4}\right)N\left(\sum_{i=1}^{N} t_i^{\,2}\right)\right)$$

This is a bit easier to look at if we make a few substitutions.

$$\mathrm{subs}\left(\mathrm{seq}\left(\sum_{i=1}^{N} t_i^{\,k}=T_k,\ k=1\,..\,4\right),\ \mathrm{seq}\left(\sum_{i=1}^{N} y_i\, t_i^{\,k}=S_k,\ k=0\,..\,2\right),\ sols\right)$$

$$\left[A=\frac{S_0\,T_3^{\,2}-S_0\,T_4\,T_2+T_1\,S_1\,T_4-T_1\,T_3\,S_2-T_2\,T_3\,S_1+T_2^{\,2}\,S_2}{-2\,T_1\,T_2\,T_3+N\,T_3^{\,2}+T_2^{\,3}+T_4\,T_1^{\,2}-T_4\,N\,T_2},\right.$$

$$B=\frac{T_4\,S_0\,T_1-T_1\,S_2\,T_2-S_1\,N\,T_4-S_0\,T_2\,T_3+N\,T_3\,S_2+T_2^{\,2}\,S_1}{-2\,T_1\,T_2\,T_3+N\,T_3^{\,2}+T_2^{\,3}+T_4\,T_1^{\,2}-T_4\,N\,T_2},$$

$$\left.C=\frac{-T_1\,T_2\,S_1+N\,T_3\,S_1-T_3\,S_0\,T_1+S_2\,T_1^{\,2}-S_2\,N\,T_2+T_2^{\,2}\,S_0}{-2\,T_1\,T_2\,T_3+N\,T_3^{\,2}+T_2^{\,3}+T_4\,T_1^{\,2}-T_4\,N\,T_2}\right]$$

Collecting on the mixed terms S_k, we have the result

$\mathrm{collect}(\%,\ [\mathrm{seq}(S_k,\ k=0\,..\,2),\ N],\ factor)$

for p **in** $\%$ **do** $\mathrm{print}(p)$ **od**

$$A = \frac{\left(T_3^{\,2} - T_4\,T_2\right)S_0}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{(-T_3\,T_2 + T_1\,T_4)\,S_1}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{\left(T_2^{\,2} - T_1\,T_3\right)S_2}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}}$$

$$B = \frac{(-T_3\,T_2 + T_1\,T_4)\,S_0}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{\left(-N\,T_4 + T_2^{\,2}\right)S_1}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{(N\,T_3 - T_1\,T_2)\,S_2}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}}$$

$$C = \frac{\left(T_2^{\,2} - T_1\,T_3\right)S_0}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{(N\,T_3 - T_1\,T_2)\,S_1}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}} + \frac{\left(T_1^{\,2} - N\,T_2\right)S_2}{\left(T_3^{\,2} - T_4\,T_2\right)N - 2\,T_1\,T_2\,T_3 + T_2^{\,3} + T_4\,T_1^{\,2}}$$

3. An Automated Linear Least Squares Solution Generator

Here is a procedure that automates the process outlined in the example of the previous section.

#----------

```
# model must be of the form Y = F(t[k],P), where P are parameters
# and t is the independent variable.  The sumindex subscript [k] MUST
# be attached to t in the model equation.  For example,
#    leastsqrs( y=a+b*x[k], x, k, [a,b] );
# Also valid is
#    leastsqrs( y[k]=a+b*x[k], x, k, [a,b] );
#-------------------------------------------------------------------
leastsqrs := proc( model::`=`, indepvar::name,
                   sumindex::name, params::list )

  local chi2, y, i, t, partials, k, eqs, solset, T, Y,
        replace_denominators, sums, YTsums, n, p, nmax, N,
        Tcount, Scount, ispoly, Delta, delta, tmp, goodsols;
  global _subslist, time0;

  if not has(model,sumindex) then
    ERROR("model must be of the form Y=F(t[i],P)");
  fi;

  time0 := time();

  i := sumindex;
  t := indepvar;
  if not type(t,indexed) then
    t := t[i];
  fi;
  y := lhs(model);
  if not type(y,indexed) then
    y := y[i];
  fi;
  chi2 := Sum((y-rhs(model))^2,i=1..N);
  if type(rhs(model),polynom) then
    ispoly := true;
  else
    ispoly := false;
  fi;
  debug_print(procname,"i,t,y,ispoly,chi2",4,i,t,y,ispoly,chi2);

  #-----------------------
  # calculate the partials
  #-----------------------
  debug_print(procname,"Forming the normal equations...",3);
  partials := [];
  for k from 1 to nops(params) do
    partials := [ op(partials), diff(chi2,params[k])=0 ];
  od;
  if printlevel >= 4 then
    debug_print(procname,"partials",4);
    for p in partials do
      print(p);
    od;
  fi;

  #-------------------------------------------
  # form the summation substitutions variables
  #-------------------------------------------
  sums := convert( select( (x,y)->op(0,x)=y,
```

```
                        indets(value(expand(partials)),function), sum ),
                 list );
#degree() can't handle summation subscripts, so remove them
YTsums    := subs( t=T, y=Y, sums );
_subslist := [];
nmax      := 0;
debug_print(procname,"summation terms",4,sums);
debug_print(procname,"YT terms",5,YTsums);
Tcount := 0;
Scount := 0;
for k from 1 to nops(YTsums) do
  if ispoly then
    select(has,[op(YTsums[k])],T);
    if %=[] then
      n := 0;
    else
      n := degree( op(%), T );
    fi;
    nmax := max(n,nmax);
  fi;
  if has(YTsums[k],Y) then
    if not ispoly then
      n := Scount;
      nmax := max(n,nmax);
      Scount := Scount + 1;
    fi;
    _subslist := [op(_subslist),sums[k]='S'[n]];
  else
    if not ispoly then
      n := Tcount;
      Tcount := Tcount + 1;
    fi;
    _subslist := [op(_subslist),sums[k]='T'[n]];
  fi;
od;

#-----------------------------------------------------
# substitute for the summation terms in the partials
#-----------------------------------------------------
partials := collect( eval( value(expand(partials)), _subslist ),
                     [seq(S[k],k=1..nmax),N], factor );

#put substitution list into form suitable for putting the sums back
_subslist := map( (x)->rhs(x)=lhs(x), _subslist );

#--------------------------
# form the normal equations
#--------------------------
eqs := collect( partials, [op(params),sum], factor );
if printlevel >= 1 then
  debug_print(procname,"normal equations",1);
  for p in eqs do
    print(p);
  od;
fi;

#---------------------------
```

```
# solve the normal equations
#---------------------------
debug_print(procname,"Solving the normal equations...",1);
_EnvAllSolutions := true;
map( allvalues, [solve( convert(eqs,set), convert(params,set) )] );
map( sortsols, %, params );
if nops(%)=1 then
  solset := op(%);
else
  solset := %;
  debug_print(procname,"There are ".(nops(solset))." solutions",1);
fi;
debug_print(procname,"raw solution(s)",4,solset);

#----------------------------------------------------
# replace the denominators with another substitution
#----------------------------------------------------

#first define a workhorse procedure
replace_denominators := proc( expr, N, nmax, S, Delta )

  local k, p, d;
  global _subslist;

  #---------------------------------
  # determine the common denominator
  #---------------------------------
  #grab and filter denominators from solution parameter expressions
  map( denom, { seq(op(rhs(expr[k])),k=1..nops(expr)) } );
  map( (x)->if type(x,{integer,fraction}) then 0 else x fi, % );
  d := % minus {0};
  if nops(d) > 1 then
    debug_print(procname,"denominator set",0,d);
    ERROR("Unable to determine a unique denominator!");
  elif d={} then
    debug_print(procname,"no denominator for solution",2,expr);
    RETURN(expr);
  else
    d := op(d);
  fi;

  #--------------------------------------
  # make the substitution in the solution
  #--------------------------------------
  _subslist := [ Delta=eval(d,_subslist), op(_subslist) ];
  collect( algsubs(d=Delta,expr), [Delta,seq(S[k],k=0..nmax),N], factor );
end;

#now make the substitutions
if type(solset,list(list)) then    #multiple solutions
  for k from 1 to nops(solset) do
    subsop(k=replace_denominators(solset[k],N,nmax,S,Delta),solset);
    solset    := subs( Delta=delta[k], % );
    _subslist := subs( Delta=delta[k], _subslist );
  od;
else                               #only one solution
  solset := replace_denominators(solset,N,nmax,S,Delta);
```

```
  fi;

  debug_print(procname,"substitution list",2,_subslist);

  #----------------------
  # check the solution(s)
  #----------------------
  if type(solset,list(list)) then  #multiple solutions
    debug_print(procname,"Verifying the ".(nops(solset))." solutions...",1);
    goodsols := [];
    for k from 1 to nops(solset) do
      tmp := expand(subs(_subslist,eval(eqs,solset[k])));
      if not type(tmp,list(0=0)) then
        debug_print(procname,"Solution ".k." is invalid!",0,solset[k]);
        debug_print(procname,"Solution ".k." substituted into normal eqs",0,eqs);
        debug_print(procname,"Throwing out solution ".k,0);
      else
        goodsols := [op(goodsols),solset[k]];
      fi;
    od;
    solset := goodsols;
    if nops(solset)=1 then
      solset := op(solset);
    fi;
  else                              #single solution
    debug_print(procname,"Verifying the solution...",1);
    eqs := factor(expand(subs(_subslist,eval(eqs,solset))));
    if not type(eqs,list(0=0)) then
      debug_print(procname,"Solution substituted into normal eqs",0,eqs);
      ERROR("Invalid solution!");
    fi;
  fi;

  if time()-time0 > 10 then
    debug_print(procname,"Done!",1);
  fi;
  solset;
end:
```

4. Examples

Polynomial of degree 1

We start with simple linear regression to check the program. We set the procedure's "chattiness" to maximum verbosity. The solution appears as the last bracketed expression.

$printlevel := 4$

$\text{leastsqrs}(y = A + B\, t_i, t, i, [A, B])$

```
leastsqrs[0]: i,t,y,ispoly,chi2
```

$$i, t_i, y_i, true, \sum_{i=1}^{N} (y_i - A - B\, t_i)^2$$

```
leastsqrs[0]: Forming the normal equations...
leastsqrs[0]: partials
```

$$\sum_{i=1}^{N} (-2\, y_i + 2\, A + 2\, B\, t_i) = 0$$

$$\sum_{i=1}^{N} (-2\, (y_i - A - B\, t_i)\, t_i) = 0$$

```
leastsqrs[0]: summation terms
```

$$\left[\sum_{i=1}^{N} y_i, \sum_{i=1}^{N} t_i\, y_i, \sum_{i=1}^{N} t_i, \sum_{i=1}^{N} t_i^{\,2}\right]$$

```
leastsqrs[0]: normal equations
```

$$-2\, S_0 + 2\, A\, N + 2\, B\, T_1 = 0$$

$$-2\, S_1 + 2\, A\, T_1 + 2\, B\, T_2 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[0]: raw solution(s)
```

$$\left[A = \frac{S_0\, T_2 - T_1\, S_1}{-T_1^{\,2} + T_2\, N}, B = -\frac{-S_1\, N + T_1\, S_0}{-T_1^{\,2} + T_2\, N}\right]$$

```
leastsqrs[0]: substitution list
```

$$\left[\Delta = -\left(\sum_{i=1}^{N} t_i\right)^2 + \left(\sum_{i=1}^{N} t_i^{\,2}\right) N, S_0 = \sum_{i=1}^{N} y_i, S_1 = \sum_{i=1}^{N} t_i\, y_i, T_1 = \sum_{i=1}^{N} t_i, T_2 = \sum_{i=1}^{N} t_i^{\,2}\right]$$

```
leastsqrs[0]: Verifying the solution...
```

$$\left[A = \frac{S_0\, T_2 - T_1\, S_1}{\Delta}, B = \frac{S_1\, N - T_1\, S_0}{\Delta}\right]$$

The substitutions are stored in the global variable *_subslist*. Hence, the full solution is

eval(*%*, *_subslist*)

$$\left[A = \frac{\left(\sum_{i=1}^{N} y_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right) - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i\, y_i\right)}{-\left(\sum_{i=1}^{N} t_i\right)^2 + \left(\sum_{i=1}^{N} t_i^{\,2}\right) N}, B = \frac{\left(\sum_{i=1}^{N} t_i\, y_i\right) N - \left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} y_i\right)}{-\left(\sum_{i=1}^{N} t_i\right)^2 + \left(\sum_{i=1}^{N} t_i^{\,2}\right) N}\right]$$

We can convert the solution into an optimized Maple procedure. This procedure can then be used to evaluate the model with a set of data. It is easily saved to disk for conversion into a C or fortran procedure — also quite easy.

tmp := array(map($x \rightarrow$ rhs(x), *%*))

```
foo := optimize(makeproc(tmp, parameters = [N, y, t]))
```

```
foo := proc(N, y, t)
local t8, t11, t4, t5, t1, tmp, t2;
    tmp := array(1 .. 2);
    t1 := sum(y[i], i = 1 .. N);
    t2 := sum(t[i]^2, i = 1 .. N);
    t4 := sum(t[i], i = 1 .. N);
    t5 := sum(t[i]*y[i], i = 1 .. N);
    t8 := t4^2;
    t11 := 1 / (-t8 + t2*N);
    tmp[1] := (t1*t2 - t4*t5)*t11;
    tmp[2] := (t5*N - t4*t1)*t11;
    tmp
end
```

```
writeto("d:/Maple/test.p")
print(foo)
writeto(terminal)
```

This is simple and straightforward in this example. Later on, we will see that the solutions can get rather unwieldy, but that the optimized procedures created from them are still relatively simple.

Polynomial of degree 2

```
printlevel := 2
```

$$\text{leastsqrs}\left(Y = A + B\, t_i + C\, t_i^{\,2},\ t,\ i,\ [A, B, C]\right)$$

```
leastsqrs[0]: normal equations
```

$$-2\, S_0 + 2\, A\, N + 2\, B\, T_1 + 2\, C\, T_2 = 0$$

$$-2\, S_1 + 2\, A\, T_1 + 2\, B\, T_2 + 2\, C\, T_3 = 0$$

$$-2\, S_2 + 2\, A\, T_2 + 2\, B\, T_3 + 2\, C\, T_4 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[0]: substitution list
```

$$\left[\Delta = -2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right) + \left(\sum_{i=1}^{N} t_i^{\,3}\right)^2 N + \left(\sum_{i=1}^{N} t_i^{\,2}\right)^3 - \left(\sum_{i=1}^{N} t_i^{\,4}\right) N \left(\sum_{i=1}^{N} t_i^{\,2}\right) + \left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i\right)^2,\ S_0 = \sum_{i=1}^{N} Y_i,\ T_1 = \sum_{i=1}^{N} t_i,\ T_2 = \sum_{i=1}^{N} t_i^{\,2},\ S_1 = \sum_{i=1}^{N} t_i\, Y_i,\ T_3 = \sum_{i=1}^{N} t_i^{\,3},\right.$$

$$S_2 = \sum_{i=1}^{N} t_i^{\,2} Y_i,\ T_4 = \sum_{i=1}^{N} t_i^{\,4} \Bigg]$$

```
leastsqrs[0]: Verifying the solution...
```

$$\Bigg[A = \frac{\left(T_3^{\,2} - T_4 T_2\right) S_0 + (-T_2 T_3 + T_1 T_4) S_1 + \left(-T_3 T_1 + T_2^{\,2}\right) S_2}{\Delta},$$

$$B = \frac{(-T_2 T_3 + T_1 T_4) S_0 + \left(T_2^{\,2} - T_4 N\right) S_1 + (-T_1 T_2 + T_3 N) S_2}{\Delta},$$

$$C = \frac{\left(-T_3 T_1 + T_2^{\,2}\right) S_0 + (-T_1 T_2 + T_3 N) S_1 + \left(-T_2 N + T_1^{\,2}\right) S_2}{\Delta} \Bigg]$$

```
tmp := array(map(x -> rhs(x), eval(%, _subslist)))
optimize(makeproc(tmp, parameters = [N, Y, t]))
```

```
proc(N, Y, t)
local t7, t26, t31, t12, t16, t28, t17, t13, t21, t18, t36, t4, t10, tmp, t3, t1, t2;
    tmp := array(1 .. 3);
    t1 := sum(t[i]^3, i = 1 .. N);
    t2 := t1^2;
    t3 := sum(t[i]^4, i = 1 .. N);
    t4 := sum(t[i]^2, i = 1 .. N);
    t7 := sum(Y[i], i = 1 .. N);
    t10 := sum(t[i], i = 1 .. N);
    t12 := - t4*t1 + t10*t3;
    t13 := sum(t[i]*Y[i], i = 1 .. N);
    t16 := t4^2;
    t17 := - t1*t10 + t16;
    t18 := sum(t[i]^2*Y[i], i = 1 .. N);
    t21 := t10*t4;
    t26 := t3*N;
    t28 := t10^2;
    t31 := 1 / (-2*t21*t1 + t2*N + t16*t4 - t26*t4 + t3*t28);
    tmp[1] := ((t2 - t3*t4)*t7 + t12*t13 + t17*t18)*t31;
    t36 := -t21 + t1*N;
    tmp[2] := (t12*t7 + (t16 - t26)*t13 + t36*t18)*t31;
    tmp[3] := (t17*t7 + t36*t13 + (- t4*N + t28)*t18)*t31;
    tmp
```

end

Polynomial of degree 3

In this example, we will not show the solution, since it is a bit unwieldy.

leastsqrs$\left(Y = A + B\, t_i + C\, t_i^{\,2} + \mathrm{D}\, t_i^{\,3},\ t,\ i,\ [A,\ B,\ C,\ \mathrm{D}]\right)$

```
leastsqrs[0]: normal equations
```

$$-2\, S_0 + 2\, A\, N + 2\, B\, T_1 + 2\, C\, T_2 + 2\, \mathrm{D}\, T_3 = 0$$

$$-2\, S_1 + 2\, A\, T_1 + 2\, B\, T_2 + 2\, C\, T_3 + 2\, \mathrm{D}\, T_4 = 0$$

$$-2\, S_2 + 2\, A\, T_2 + 2\, B\, T_3 + 2\, C\, T_4 + 2\, \mathrm{D}\, T_5 = 0$$

$$-2\, S_3 + 2\, A\, T_3 + 2\, B\, T_4 + 2\, C\, T_5 + 2\, \mathrm{D}\, T_6 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[1]: substitution list
```

$$\left[\Delta = \left(\sum_{i=1}^{N} t_i^{\,6}\right)\left(\sum_{i=1}^{N} t_i^{\,2}\right)^3 - 2\left(\sum_{i=1}^{N} t_i^{\,2}\right)^2\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i^{\,5}\right) - \left(\sum_{i=1}^{N} t_i^{\,2}\right)^2\left(\sum_{i=1}^{N} t_i^{\,4}\right)^2 + \left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,5}\right)^2 N - \left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,6}\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right) N + 2\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right)\left(\sum_{i=1}^{N} t_i^{\,5}\right) + 3\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2\left(\sum_{i=1}^{N} t_i^{\,4}\right) - 2\left(\sum_{i=1}^{N} t_i^{\,2}\right)\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,6}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right) + \left(\sum_{i=1}^{N} t_i^{\,4}\right)^3 N + \left(\sum_{i=1}^{N} t_i^{\,6}\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2 N - 2\left(\sum_{i=1}^{N} t_i^{\,4}\right) N \left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i^{\,5}\right) - 2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right)^2 + 2\left(\sum_{i=1}^{N} t_i\right)\left(\sum_{i=1}^{N} t_i^{\,3}\right)^2\left(\sum_{i=1}^{N} t_i^{\,5}\right) + \left(\sum_{i=1}^{N} t_i\right)^2\left(\sum_{i=1}^{N} t_i^{\,6}\right)\left(\sum_{i=1}^{N} t_i^{\,4}\right) - \left(\sum_{i=1}^{N} t_i\right)^2\left(\sum_{i=1}^{N} t_i^{\,5}\right)^2 - \left(\sum_{i=1}^{N} t_i^{\,3}\right)^4,\ T_6 = \sum_{i=1}^{N} t_i^{\,6},\ T_5 = \sum_{i=1}^{N} t_i^{\,5},\ S_0 = \sum_{i=1}^{N} Y_i,\ T_2 = \sum_{i=1}^{N} t_i^{\,2},\ T_1 = \sum_{i=1}^{N} t_i,\ T_3 = \sum_{i=1}^{N} t_i^{\,3},\ S_1 = \sum_{i=1}^{N} t_i\, Y_i,\ S_3 = \sum_{i=1}^{N} t_i^{\,3}\, Y_i,\ T_4 = \sum_{i=1}^{N} t_i^{\,4},\ S_2 = \sum_{i=1}^{N} t_i^{\,2}\, Y_i\right]$$

```
leastsqrs[2]: Verifying the solution...
```

map($cost$, %)

$$[21\ \textit{additions} + 50\ \textit{multiplications} + \textit{divisions} + \textit{assignments},$$
$$22\ \textit{additions} + 48\ \textit{multiplications} + \textit{divisions} + \textit{assignments},$$
$$22\ \textit{additions} + 48\ \textit{multiplications} + \textit{divisions} + \textit{assignments},$$
$$21\ \textit{additions} + 47\ \textit{multiplications} + \textit{divisions} + \textit{assignments}]$$

The following commands produce an optimized procedure, which we will not show here.

$tmp := \mathrm{array}(\mathrm{map}(x \rightarrow \mathrm{rhs}(x), \mathrm{eval}(\%\%, _subslist)))$

$\mathrm{optimize}(\mathrm{makeproc}(tmp, parameters = [N, Y, t]))$

Sinusoid — Representation 1

$printlevel := 1$

$\mathrm{leastsqrs}(Y = A\cos(2\pi\nu t_i + \phi), t, i, [A, \phi])$

```
leastsqrs[0]: normal equations
```

$$(8\cos(\phi)^2 T_0 - 8\cos(\phi)^2 T_1 - 16\cos(\phi)\sin(\phi) T_2 + 2\cos(\phi)^2 N + 8\cos(\phi)\sin(\phi) T_3 + 8\sin(\phi)^2 T_4) A - 4\cos(\phi) S_0 + 2\cos(\phi) S_1 + 4\sin(\phi) S_2 = 0$$

$$(-2\cos(\phi)\sin(\phi) N - 8\cos(\phi)\sin(\phi) T_0 + 4\cos(\phi)^2 T_3 - 8\cos(\phi)^2 T_2 + 8\cos(\phi)\sin(\phi) T_4 + 8\cos(\phi)\sin(\phi) T_1 - 4\sin(\phi)^2 T_3 + 8\sin(\phi)^2 T_2) A^2 + (-2\sin(\phi) S_1 + 4\cos(\phi) S_2 + 4\sin(\phi) S_0) A = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[2]: There are 3 solutions
leastsqrs[4]: Verifying the 3 solutions...
leastsqrs[18]: Solution 2 is invalid!
```

$$\Big[A = \frac{1}{2}\mathrm{sqrt}\Big(16 T_2 S_2 S_1 T_4 + 16 T_3 S_2 S_0 T_4 - 8 T_3 S_2 S_1 T_4 - 32 T_2 S_2 S_0 T_4 - 16 T_2 S_2^2 T_3 - 16 S_0 T_4^2 S_1 + 16 T_2^2 S_2^2 - 4 T_2 S_1^2 T_3 - 4 T_3^2 S_0 S_1 + 16 T_2 S_0 T_3 S_1 + 8 S_2 T_1 T_3 S_1 - 8 S_2 T_0 T_3 S_1 + 32 S_2 T_1 T_2 S_0 - 16 S_2 T_1 T_2 S_1 - 16 S_2 T_1 T_3 S_0 - 16 T_2^2 S_0 S_1 - 2 S_2 N T_3 S_1 - 32 S_2 T_0 T_2 S_0 + 16 S_2 T_0 T_2 S_1 + 16 S_2 T_0 T_3 S_0 + 4 S_2 N T_2 S_1 + 4 S_2 N T_3 S_0 - 16 T_2 S_0^2 T_3 + 8 S_2^2 N T_0 - 8 S_2^2 N T_1 - 32 S_2^2 T_0 T_1 + S_2^2 N^2 + 16 S_2^2 T_0^2 + 16 S_2^2 T_1^2 + 16 T_2^2 S_0^2 + 4 T_2^2 S_1^2 + 4 T_3^2 S_0^2 + T_3^2 S_1^2 - 8 S_2 N T_2 S_0$$

$$+4\,T_3^{\,2}\,S_2^{\,2}+16\,S_0^{\,2}\,T_4^{\,2}+4\,S_1^{\,2}\,T_4^{\,2}\Big)\Big/\,\delta_2,\ \phi=\arctan\Big(($$

$$-S_2\,N-4\,S_2\,T_0+4\,S_2\,T_1+4\,T_2\,S_0-2\,T_2\,S_1-2\,T_3\,S_0+T_3\,S_1)\Big/\,\mathrm{sqrt}\Big(16\,T_2\,S_2\,S_1\,T_4$$

$$+16\,T_3\,S_2\,S_0\,T_4-8\,T_3\,S_2\,S_1\,T_4-32\,T_2\,S_2\,S_0\,T_4-16\,T_2\,S_2^{\,2}\,T_3-16\,S_0\,T_4^{\,2}\,S_1+16\,T_2^{\,2}\,S_2^{\,2}$$

$$-4\,T_2\,S_1^{\,2}\,T_3-4\,T_3^{\,2}\,S_0\,S_1+16\,T_2\,S_0\,T_3\,S_1+8\,S_2\,T_1\,T_3\,S_1-8\,S_2\,T_0\,T_3\,S_1$$

$$+32\,S_2\,T_1\,T_2\,S_0-16\,S_2\,T_1\,T_2\,S_1-16\,S_2\,T_1\,T_3\,S_0-16\,T_2^{\,2}\,S_0\,S_1-2\,S_2\,N\,T_3\,S_1$$

$$-32\,S_2\,T_0\,T_2\,S_0+16\,S_2\,T_0\,T_2\,S_1+16\,S_2\,T_0\,T_3\,S_0+4\,S_2\,N\,T_2\,S_1+4\,S_2\,N\,T_3\,S_0$$

$$-16\,T_2\,S_0^{\,2}\,T_3+8\,S_2^{\,2}\,N\,T_0-8\,S_2^{\,2}\,N\,T_1-32\,S_2^{\,2}\,T_0\,T_1+S_2^{\,2}\,N^2+16\,S_2^{\,2}\,T_0^{\,2}+16\,S_2^{\,2}\,T_1^{\,2}$$

$$+16\,T_2^{\,2}\,S_0^{\,2}+4\,T_2^{\,2}\,S_1^{\,2}+4\,T_3^{\,2}\,S_0^{\,2}+T_3^{\,2}\,S_1^{\,2}-8\,S_2\,N\,T_2\,S_0+4\,T_3^{\,2}\,S_2^{\,2}+16\,S_0^{\,2}\,T_4^{\,2}$$

$$+4\,S_1^{\,2}\,T_4^{\,2}\Big),\ 2\,(-2\,T_2\,S_2+T_3\,S_2+2\,S_0\,T_4-S_1\,T_4)\Big/\,\mathrm{sqrt}\Big(16\,T_2\,S_2\,S_1\,T_4+16\,T_3\,S_2\,S_0\,T_4$$

$$-8\,T_3\,S_2\,S_1\,T_4-32\,T_2\,S_2\,S_0\,T_4-16\,T_2\,S_2^{\,2}\,T_3-16\,S_0\,T_4^{\,2}\,S_1+16\,T_2^{\,2}\,S_2^{\,2}-4\,T_2\,S_1^{\,2}\,T_3$$

$$-4\,T_3^{\,2}\,S_0\,S_1+16\,T_2\,S_0\,T_3\,S_1+8\,S_2\,T_1\,T_3\,S_1-8\,S_2\,T_0\,T_3\,S_1+32\,S_2\,T_1\,T_2\,S_0$$

$$-16\,S_2\,T_1\,T_2\,S_1-16\,S_2\,T_1\,T_3\,S_0-16\,T_2^{\,2}\,S_0\,S_1-2\,S_2\,N\,T_3\,S_1-32\,S_2\,T_0\,T_2\,S_0$$

$$+16\,S_2\,T_0\,T_2\,S_1+16\,S_2\,T_0\,T_3\,S_0+4\,S_2\,N\,T_2\,S_1+4\,S_2\,N\,T_3\,S_0-16\,T_2\,S_0^{\,2}\,T_3+8\,S_2^{\,2}\,N\,T_0$$

$$-8\,S_2^{\,2}\,N\,T_1-32\,S_2^{\,2}\,T_0\,T_1+S_2^{\,2}\,N^2+16\,S_2^{\,2}\,T_0^{\,2}+16\,S_2^{\,2}\,T_1^{\,2}+16\,T_2^{\,2}\,S_0^{\,2}+4\,T_2^{\,2}\,S_1^{\,2}$$

$$+4\,T_3^{\,2}\,S_0^{\,2}+T_3^{\,2}\,S_1^{\,2}-8\,S_2\,N\,T_2\,S_0+4\,T_3^{\,2}\,S_2^{\,2}+16\,S_0^{\,2}\,T_4^{\,2}+4\,S_1^{\,2}\,T_4^{\,2}\Big)\Big)\Big)+2\,\pi_Z\Big]$$

```
leastsqrs[18]: Solution 2 substituted into normal eqs
```

$$[(8\cos(\phi)^2\,T_0-8\cos(\phi)^2\,T_1-16\cos(\phi)\sin(\phi)\,T_2+2\cos(\phi)^2\,N+8\cos(\phi)\sin(\phi)\,T_3$$

$$+8\sin(\phi)^2\,T_4)\,A-4\cos(\phi)\,S_0+2\cos(\phi)\,S_1+4\sin(\phi)\,S_2=0,\ (-2\cos(\phi)\sin(\phi)\,N$$

$$-8\cos(\phi)\sin(\phi)\,T_0+4\cos(\phi)^2\,T_3-8\cos(\phi)^2\,T_2+8\cos(\phi)\sin(\phi)\,T_4+8\cos(\phi)\sin(\phi)\,T_1$$

$$-4\sin(\phi)^2\,T_3+8\sin(\phi)^2\,T_2)\,A^2+(-2\sin(\phi)\,S_1+4\cos(\phi)\,S_2+4\sin(\phi)\,S_0)\,A=0]$$

```
leastsqrs[18]: Throwing out solution 2
leastsqrs[31]: Solution 3 is invalid!
```

$$\Big[A = -\frac{1}{2}\,\mathrm{sqrt}\Big(16\,T_2\,S_2\,S_1\,T_4 + 16\,T_3\,S_2\,S_0\,T_4 - 8\,T_3\,S_2\,S_1\,T_4 - 32\,T_2\,S_2\,S_0\,T_4 - 16\,T_2\,S_2^{\,2}\,T_3$$

$$- 16\,S_0\,T_4^{\,2}\,S_1 + 16\,T_2^{\,2}\,S_2^{\,2} - 4\,T_2\,S_1^{\,2}\,T_3 - 4\,T_3^{\,2}\,S_0\,S_1 + 16\,T_2\,S_0\,T_3\,S_1 + 8\,S_2\,T_1\,T_3\,S_1$$

$$- 8\,S_2\,T_0\,T_3\,S_1 + 32\,S_2\,T_1\,T_2\,S_0 - 16\,S_2\,T_1\,T_2\,S_1 - 16\,S_2\,T_1\,T_3\,S_0 - 16\,T_2^{\,2}\,S_0\,S_1$$

$$- 2\,S_2\,N\,T_3\,S_1 - 32\,S_2\,T_0\,T_2\,S_0 + 16\,S_2\,T_0\,T_2\,S_1 + 16\,S_2\,T_0\,T_3\,S_0 + 4\,S_2\,N\,T_2\,S_1$$

$$+ 4\,S_2\,N\,T_3\,S_0 - 16\,T_2\,S_0^{\,2}\,T_3 + 8\,S_2^{\,2}\,N\,T_0 - 8\,S_2^{\,2}\,N\,T_1 - 32\,S_2^{\,2}\,T_0\,T_1 + S_2^{\,2}\,N^2$$

$$+ 16\,S_2^{\,2}\,T_0^{\,2} + 16\,S_2^{\,2}\,T_1^{\,2} + 16\,T_2^{\,2}\,S_0^{\,2} + 4\,T_2^{\,2}\,S_1^{\,2} + 4\,T_3^{\,2}\,S_0^{\,2} + T_3^{\,2}\,S_1^{\,2} - 8\,S_2\,N\,T_2\,S_0$$

$$+ 4\,T_3^{\,2}\,S_2^{\,2} + 16\,S_0^{\,2}\,T_4^{\,2} + 4\,S_1^{\,2}\,T_4^{\,2}\Big) \Big/ \delta_3,\ \phi = \arctan\Big(-\,($$

$$-S_2\,N - 4\,S_2\,T_0 + 4\,S_2\,T_1 + 4\,T_2\,S_0 - 2\,T_2\,S_1 - 2\,T_3\,S_0 + T_3\,S_1) \Big/ \mathrm{sqrt}\Big(16\,T_2\,S_2\,S_1\,T_4$$

$$+ 16\,T_3\,S_2\,S_0\,T_4 - 8\,T_3\,S_2\,S_1\,T_4 - 32\,T_2\,S_2\,S_0\,T_4 - 16\,T_2\,S_2^{\,2}\,T_3 - 16\,S_0\,T_4^{\,2}\,S_1 + 16\,T_2^{\,2}\,S_2^{\,2}$$

$$- 4\,T_2\,S_1^{\,2}\,T_3 - 4\,T_3^{\,2}\,S_0\,S_1 + 16\,T_2\,S_0\,T_3\,S_1 + 8\,S_2\,T_1\,T_3\,S_1 - 8\,S_2\,T_0\,T_3\,S_1$$

$$+ 32\,S_2\,T_1\,T_2\,S_0 - 16\,S_2\,T_1\,T_2\,S_1 - 16\,S_2\,T_1\,T_3\,S_0 - 16\,T_2^{\,2}\,S_0\,S_1 - 2\,S_2\,N\,T_3\,S_1$$

$$- 32\,S_2\,T_0\,T_2\,S_0 + 16\,S_2\,T_0\,T_2\,S_1 + 16\,S_2\,T_0\,T_3\,S_0 + 4\,S_2\,N\,T_2\,S_1 + 4\,S_2\,N\,T_3\,S_0$$

$$- 16\,T_2\,S_0^{\,2}\,T_3 + 8\,S_2^{\,2}\,N\,T_0 - 8\,S_2^{\,2}\,N\,T_1 - 32\,S_2^{\,2}\,T_0\,T_1 + S_2^{\,2}\,N^2 + 16\,S_2^{\,2}\,T_0^{\,2} + 16\,S_2^{\,2}\,T_1^{\,2}$$

$$+ 16\,T_2^{\,2}\,S_0^{\,2} + 4\,T_2^{\,2}\,S_1^{\,2} + 4\,T_3^{\,2}\,S_0^{\,2} + T_3^{\,2}\,S_1^{\,2} - 8\,S_2\,N\,T_2\,S_0 + 4\,T_3^{\,2}\,S_2^{\,2} + 16\,S_0^{\,2}\,T_4^{\,2}$$

$$+ 4\,S_1^{\,2}\,T_4^{\,2}\Big) - 2\,(-2\,T_2\,S_2 + T_3\,S_2 + 2\,S_0\,T_4 - S_1\,T_4) \Big/ \mathrm{sqrt}\Big(16\,T_2\,S_2\,S_1\,T_4$$

$$+ 16\,T_3\,S_2\,S_0\,T_4 - 8\,T_3\,S_2\,S_1\,T_4 - 32\,T_2\,S_2\,S_0\,T_4 - 16\,T_2\,S_2^{\,2}\,T_3 - 16\,S_0\,T_4^{\,2}\,S_1 + 16\,T_2^{\,2}\,S_2^{\,2}$$

$$- 4\,T_2\,S_1^{\,2}\,T_3 - 4\,T_3^{\,2}\,S_0\,S_1 + 16\,T_2\,S_0\,T_3\,S_1 + 8\,S_2\,T_1\,T_3\,S_1 - 8\,S_2\,T_0\,T_3\,S_1$$

$$+ 32\,S_2\,T_1\,T_2\,S_0 - 16\,S_2\,T_1\,T_2\,S_1 - 16\,S_2\,T_1\,T_3\,S_0 - 16\,T_2^{\,2}\,S_0\,S_1 - 2\,S_2\,N\,T_3\,S_1$$

$$- 32\,S_2\,T_0\,T_2\,S_0 + 16\,S_2\,T_0\,T_2\,S_1 + 16\,S_2\,T_0\,T_3\,S_0 + 4\,S_2\,N\,T_2\,S_1 + 4\,S_2\,N\,T_3\,S_0$$

$$- 16\,T_2\,S_0^{\,2}\,T_3 + 8\,S_2^{\,2}\,N\,T_0 - 8\,S_2^{\,2}\,N\,T_1 - 32\,S_2^{\,2}\,T_0\,T_1 + S_2^{\,2}\,N^2 + 16\,S_2^{\,2}\,T_0^{\,2} + 16\,S_2^{\,2}\,T_1^{\,2}$$

$$+16\,T_2^{\,2}\,S_0^{\,2}+4\,T_2^{\,2}\,S_1^{\,2}+4\,T_3^{\,2}\,S_0^{\,2}+T_3^{\,2}\,S_1^{\,2}-8\,S_2\,N\,T_2\,S_0+4\,T_3^{\,2}\,S_2^{\,2}+16\,S_0^{\,2}\,T_4^{\,2}$$
$$+4\,S_1^{\,2}\,T_4^{\,2}\Big)\Big)+2\,\pi\,_Z\Big]$$

```
leastsqrs[31]: Solution 3 substituted into normal eqs
```

$$[(8\cos(\phi)^2\,T_0-8\cos(\phi)^2\,T_1-16\cos(\phi)\sin(\phi)\,T_2+2\cos(\phi)^2\,N+8\cos(\phi)\sin(\phi)\,T_3$$
$$+8\sin(\phi)^2\,T_4)\,A-4\cos(\phi)\,S_0+2\cos(\phi)\,S_1+4\sin(\phi)\,S_2=0,\,(-2\cos(\phi)\sin(\phi)\,N$$
$$-8\cos(\phi)\sin(\phi)\,T_0+4\cos(\phi)^2\,T_3-8\cos(\phi)^2\,T_2+8\cos(\phi)\sin(\phi)\,T_4+8\cos(\phi)\sin(\phi)\,T_1$$
$$-4\sin(\phi)^2\,T_3+8\sin(\phi)^2\,T_2)\,A^2+(-2\sin(\phi)\,S_1+4\cos(\phi)\,S_2+4\sin(\phi)\,S_0)\,A=0]$$

```
leastsqrs[31]: Throwing out solution 3
leastsqrs[31]: Done!
```

$$\left[A=0,\,\phi=\arctan\left(\frac{1}{2}\,\frac{2\,S_0-S_1}{S_2}\right)+\pi\,_Z\right]$$

As we can see, $Y=A\cos(2\,\pi\,\nu\,t_i+\phi)$ is not a particularly good choice of representation, since the solution involves inverse trigonometric functions and is particularly useless ($A=0$). The next, equivalent, example leads to a useful solution.

Sinusoid — Representation 2

printlevel := 2

leastsqrs($Y=A\cos(2\,\pi\,\nu\,t_i)+B\sin(2\,\pi\,\nu\,t_i),\,t,\,i,\,[A,\,B]$)

```
leastsqrs[0]: normal equations
```

$$(-8\,T_1+2\,N+8\,T_0)\,A+(-4\,T_3+8\,T_2)\,B-4\,S_2+2\,S_0=0$$

$$(-4\,T_3+8\,T_2)\,A-4\,S_1+8\,B\,T_4=0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[1]: substitution list
```

$$\left[\Delta=4\left(\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^4\right)\left(\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)^2\cos(\pi\,\nu\,t_i)^2\right)\right.$$
$$-4\left(\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^2\right)\left(\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)^2\cos(\pi\,\nu\,t_i)^2\right)+N\left(\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)^2\cos(\pi\,\nu\,t_i)^2\right)$$
$$-4\left(\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^3\sin(\pi\,\nu\,t_i)\right)^2$$

$$+4\left(\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)\cos(\pi\,\nu\,t_i)\right)\left(\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^3\sin(\pi\,\nu\,t_i)\right)$$

$$-\left(\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)\cos(\pi\,\nu\,t_i)\right)^2,\ S_0=\sum_{i=1}^{N}Y_i,\ T_0=\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^4,\ T_1=\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^2,$$

$$T_2=\sum_{i=1}^{N}\cos(\pi\,\nu\,t_i)^3\sin(\pi\,\nu\,t_i),\ T_3=\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)\cos(\pi\,\nu\,t_i),$$

$$S_1=\sum_{i=1}^{N}Y_i\sin(\pi\,\nu\,t_i)\cos(\pi\,\nu\,t_i),\ T_4=\sum_{i=1}^{N}\sin(\pi\,\nu\,t_i)^2\cos(\pi\,\nu\,t_i)^2,\ S_2=\sum_{i=1}^{N}Y_i\cos(\pi\,\nu\,t_i)^2$$

$$\Bigg]$$

```
leastsqrs[1]: Verifying the solution...
```

$$\left[A=\frac{-S_0\,T_4+(-2\,T_2+T_3)\,S_1+2\,S_2\,T_4}{\Delta},\right.$$

$$\left.B=\frac{\left(T_2-\frac{1}{2}T_3\right)S_0+\left(2\,T_0-2\,T_1+\frac{1}{2}N\right)S_1+(-2\,T_2+T_3)\,S_2}{\Delta}\right]$$

tmp := array(map($x \to$ rhs(x), eval(%, *_subslist*)))

optimize(makeproc(*tmp*, *parameters* = [*N*, *Y*, *t*]))

```
proc(N, Y, t)
local tmp, t1, t2, t8, t14, t7, t6, t4, t10, t17, t21, t27, t25;
    tmp := array(1 .. 2);
    t1 := sum(Y[i], i = 1 .. N);
    t2 := sum(sin(π*ν*t[i])^2*cos(π*ν*t[i])^2, i = 1 .. N);
    t4 := sum(cos(π*ν*t[i])^3*sin(π*ν*t[i]), i = 1 .. N);
    t6 := sum(sin(π*ν*t[i])*cos(π*ν*t[i]), i = 1 .. N);
    t7 := -2*t4 + t6;
    t8 := sum(Y[i]*sin(π*ν*t[i])*cos(π*ν*t[i]), i = 1 .. N);
    t10 := sum(Y[i]*cos(π*ν*t[i])^2, i = 1 .. N);
    t14 := sum(cos(π*ν*t[i])^4, i = 1 .. N);
    t17 := sum(cos(π*ν*t[i])^2, i = 1 .. N);
    t21 := t4^2;
    t25 := t6^2;
    t27 := 1 / (4*t2*t14 - 4*t17*t2 + N*t2 - 4*t21 + 4*t6*t4 - t25);
    tmp[1] := (- t1*t2 + t7*t8 + 2*t10*t2)*t27;
```

$tmp[2] := ((t4 - 1/2*t6)*t1 + (2*t14 - 2*t17 + 1/2*N)*t8 + t7*t10)*t27;$

tmp

end

Logarithm

leastsqrs($Y = A + B \log(t_i)$, t, i, $[A, B]$)

```
leastsqrs[0]: normal equations
```

$$-2\, S_0 + 2\, A\, N + 2\, B\, T_1 = 0$$

$$-2\, S_1 + 2\, A\, T_1 + 2\, B\, T_0 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[0]: substitution list
```

$$\left[\Delta = -\left(\sum_{i=1}^{N} \ln(t_i)\right)^2 + \left(\sum_{i=1}^{N} \ln(t_i)^2\right) N,\ S_0 = \sum_{i=1}^{N} Y_i,\ T_0 = \sum_{i=1}^{N} \ln(t_i)^2,\ T_1 = \sum_{i=1}^{N} \ln(t_i),\ S_1 = \sum_{i=1}^{N} \ln(t_i)\, Y_i\right]$$

```
leastsqrs[0]: Verifying the solution...
```

$$\left[A = \frac{S_0\, T_0 - T_1\, S_1}{\Delta},\ B = \frac{S_1\, N - T_1\, S_0}{\Delta}\right]$$

eval($\%$, $_subslist$)

$$\left[A = \frac{\left(\sum_{i=1}^{N} Y_i\right)\left(\sum_{i=1}^{N} \ln(t_i)^2\right) - \left(\sum_{i=1}^{N} \ln(t_i)\right)\left(\sum_{i=1}^{N} \ln(t_i)\, Y_i\right)}{-\left(\sum_{i=1}^{N} \ln(t_i)\right)^2 + \left(\sum_{i=1}^{N} \ln(t_i)^2\right) N},\ B = \frac{\left(\sum_{i=1}^{N} \ln(t_i)\, Y_i\right) N - \left(\sum_{i=1}^{N} \ln(t_i)\right)\left(\sum_{i=1}^{N} Y_i\right)}{-\left(\sum_{i=1}^{N} \ln(t_i)\right)^2 + \left(\sum_{i=1}^{N} \ln(t_i)^2\right) N}\right]$$

tmp := array(map($x \rightarrow$ rhs(x), eval($\%\%$, $_subslist$)))

optimize(makeproc(tmp, $parameters = [N, Y, t]$))

proc(N, Y, t)

local tmp, $t5$, $t11$, $t4$, $t8$, $t1$, $t2$;

tmp := array(1 .. 2);

```
    t1 := sum(Y[i], i = 1 .. N);
    t2 := sum(ln(t[i])^2, i = 1 .. N);
    t4 := sum(ln(t[i]), i = 1 .. N);
    t5 := sum(ln(t[i])*Y[i], i = 1 .. N);
    t8 := t4^2;
    t11 := 1 / (-t8 + t2*N);
    tmp[1] := (t2*t1 - t4*t5)*t11;
    tmp[2] := (t5*N - t1*t4)*t11;
    tmp
end
```

Exponential

$$\text{leastsqrs}\left(Y = A + B\,\mathbf{e}^{t_i}, t, i, [A, B]\right)$$

```
leastsqrs[0]: normal equations
```

$$-2\,S_0 + 2\,A\,N + 2\,B\,T_0 = 0$$

$$-2\,S_1 + 2\,A\,T_0 + 2\,B\,T_1 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[0]: substitution list
```

$$\left[\Delta = -\left(\sum_{i=1}^{N} \mathbf{e}^{t_i}\right)^2 + \left(\sum_{i=1}^{N} \left(\mathbf{e}^{t_i}\right)^2\right) N, S_0 = \sum_{i=1}^{N} Y_i, T_0 = \sum_{i=1}^{N} \mathbf{e}^{t_i}, S_1 = \sum_{i=1}^{N} \mathbf{e}^{t_i} Y_i, T_1 = \sum_{i=1}^{N} \left(\mathbf{e}^{t_i}\right)^2\right]$$

```
leastsqrs[0]: Verifying the solution...
```

$$\left[A = \frac{S_0\,T_1 - T_0\,S_1}{\Delta}, B = \frac{S_1\,N - T_0\,S_0}{\Delta}\right]$$

$$\text{eval}(\%, _subslist)$$

$$\left[A = \frac{\left(\sum_{i=1}^{N} Y_i\right)\left(\sum_{i=1}^{N} \left(\mathbf{e}^{t_i}\right)^2\right) - \left(\sum_{i=1}^{N} \mathbf{e}^{t_i}\right)\left(\sum_{i=1}^{N} \mathbf{e}^{t_i} Y_i\right)}{-\left(\sum_{i=1}^{N} \mathbf{e}^{t_i}\right)^2 + \left(\sum_{i=1}^{N} \left(\mathbf{e}^{t_i}\right)^2\right) N},\right.$$

$$B = \frac{\left(\sum_{i=1}^{N} \mathbf{e}^{t_i} Y_i\right) N - \left(\sum_{i=1}^{N} \mathbf{e}^{t_i}\right)\left(\sum_{i=1}^{N} Y_i\right)}{-\left(\sum_{i=1}^{N} \mathbf{e}^{t_i}\right)^2 + \left(\sum_{i=1}^{N} \left(\mathbf{e}^{t_i}\right)^2\right) N}$$

tmp := array(map($x \rightarrow$ rhs(x), eval(*%%*, *_subslist*)))

optimize(makeproc(*tmp*, *parameters* = [*N*, *Y*, *t*]))

```
proc(N, Y, t)
local t5, t11, t1, t2, tmp, t4, t8;
    tmp := array(1 .. 2);
    t1 := sum(Y[i], i = 1 .. N);
    t2 := sum(exp(t[i])^2, i = 1 .. N);
    t4 := sum(exp(t[i]), i = 1 .. N);
    t5 := sum(exp(t[i])*Y[i], i = 1 .. N);
    t8 := t4^2;
    t11 := 1 / (-t8 + t2*N);
    tmp[1] := (t1*t2 - t4*t5)*t11;
    tmp[2] := (t5*N - t4*t1)*t11;
    tmp
end
```

Exponentially Decaying Sinusoid

The substitution list for this case is quite long, so its output will be suppressed.

printlevel := 1

leastsqrs$\left(Y = A + \mathbf{e}^{(-t_i)} (B \cos(2 \pi \nu t_i) + C \sin(2 \pi \nu t_i)), t, i, [A, B, C]\right)$

```
leastsqrs[0]: normal equations
```

$$2 A N + (4 T_0 - 2 T_2) B - 2 S_0 + 4 C T_3 = 0$$

$$(4 T_0 - 2 T_2) A + (8 T_4 - 8 T_6 + 2 T_5) B + (-4 T_8 + 8 T_7) C - 4 S_1 + 2 S_2 = 0$$

$$4 A T_3 + (-4 T_8 + 8 T_7) B - 4 S_3 + 8 C T_1 = 0$$

```
leastsqrs[0]: Solving the normal equations...
leastsqrs[2]: Verifying the solution...
```

soln := *%*

for *p* **in** *soln* **do** print(*p*) **od**

$$A = \Big(\Big(4\,T_7^{\,2} + T_8^{\,2} - 4\,T_7\,T_8 - T_5\,T_1 - 4\,T_4\,T_1 + 4\,T_6\,T_1\Big)S_0 + (-4\,T_7\,T_3 + 4\,T_0\,T_1 + 2\,T_8\,T_3 - 2\,T_2\,T_1)\,S_1 + (T_2\,T_1 + 2\,T_7\,T_3 - T_8\,T_3 - 2\,T_0\,T_1)\,S_2 + (-4\,T_3\,T_6 + T_3\,T_5 + 4\,T_3\,T_4 - 4\,T_0\,T_7 + 2\,T_7\,T_2 + 2\,T_0\,T_8 - T_2\,T_8)\,S_3\Big)/\,\Delta$$

$$B = \Big((-2\,T_7\,T_3 + 2\,T_0\,T_1 + T_8\,T_3 - T_2\,T_1)\,S_0 + \Big(-2\,T_1\,N + 2\,T_3^{\,2}\Big)S_1 + \Big(T_1\,N - T_3^{\,2}\Big)S_2 + ((-T_8 + 2\,T_7)\,N - T_3\,(2\,T_0 - T_2))\,S_3\Big)/\,\Delta$$

$$C = \Big(\Big(-\frac{1}{2}\,T_2\,T_8 - 2\,T_0\,T_7 + T_0\,T_8 + T_7\,T_2 - 2\,T_3\,T_6 + \frac{1}{2}\,T_3\,T_5 + 2\,T_3\,T_4\Big)S_0 + ((-T_8 + 2\,T_7)\,N - T_3\,(2\,T_0 - T_2))\,S_1 + \Big(\Big(\frac{1}{2}\,T_8 - T_7\Big)N + \frac{1}{2}\,T_3\,(2\,T_0 - T_2)\Big)S_2 + \Big(\Big(2\,T_6 - \frac{1}{2}\,T_5 - 2\,T_4\Big)N + \frac{1}{2}\,(2\,T_0 - T_2)^2\Big)S_3\Big)/\,\Delta$$

The following commands produce an optimized procedure, which we will not show here.

tmp := array(map($x \to$ rhs(x), eval($soln$, $_subslist$)))

optimize(makeproc(tmp, $parameters = [N, Y, t]$))

Sinusoid with Exponentially Decaying Component

$$model := Y = A + B\cos(2\,\pi\,\nu_1\,t_i) + C\sin(2\,\pi\,\nu_1\,t_i) + \mathbf{e}^{(-t_i)}\,(\mathrm{D}\cos(2\,\pi\,\nu_2\,t_i) + E\sin(2\,\pi\,\nu_2\,t_i))$$

leastsqrs($model$, t, i, [A, B, C, D, E])

```
leastsqrs[4]: normal equations
```

$$2\,N\,A + (-2\,N + 4\,T_{11})\,B + 4\,C\,T_{12} + (4\,T_{13} - 2\,T_{15})\,\mathrm{D} - 2\,S_3 + 4\,E\,T_{18} = 0$$

$$(-2\,N + 4\,T_{11})\,A + (-8\,T_{11} + 2\,N + 8\,T_3)\,B + (8\,T_6 - 4\,T_{12})\,C + (-4\,T_{13} - 4\,T_9 + 2\,T_{15} + 8\,T_7)\,\mathrm{D} + (8\,T_{10} - 4\,T_{18})\,E + 2\,S_3 - 4\,S_5 = 0$$

$$4\,A\,T_{12} + (8\,T_6 - 4\,T_{12})\,B + 8\,C\,T_{16} + (8\,T_{17} - 4\,T_{19})\,\mathrm{D} - 4\,S_4 + 8\,E\,T_8 = 0$$

$$(4\,T_{13} - 2\,T_{15})\,A + (-4\,T_{13} - 4\,T_9 + 2\,T_{15} + 8\,T_7)\,B + (8\,T_{17} - 4\,T_{19})\,C + (8\,T_{14} - 8\,T_0 + 2\,T_1)\,\mathrm{D} + (8\,T_2 - 4\,T_4)\,E - 4\,S_0 + 2\,S_1 = 0$$

$$4\,A\,T_{18} + (8\,T_{10} - 4\,T_{18})\,B + 8\,C\,T_8 + (8\,T_2 - 4\,T_4)\,\mathrm{D} - 4\,S_2 + 8\,E\,T_5 = 0$$

```
leastsqrs[4]: Solving the normal equations...
leastsqrs[46]: Verifying the solution...
leastsqrs[569]: Done!
```

soln := %

As we can see, the solution for this model is quite complicated:

cost(*soln*)

$$2188\ \mathit{additions} + 7749\ \mathit{multiplications} + 5\ \mathit{divisions} + 5\ \mathit{assignments}$$

map(*cost*, *soln*)

$$[789\ \mathit{additions} + 2864\ \mathit{multiplications} + \mathit{divisions} + \mathit{assignments},$$
$$394\ \mathit{additions} + 1382\ \mathit{multiplications} + \mathit{divisions} + \mathit{assignments},$$
$$394\ \mathit{additions} + 1382\ \mathit{multiplications} + \mathit{divisions} + \mathit{assignments},$$
$$217\ \mathit{additions} + 739\ \mathit{multiplications} + \mathit{divisions} + \mathit{assignments},$$
$$394\ \mathit{additions} + 1382\ \mathit{multiplications} + \mathit{divisions} + \mathit{assignments}]$$

cost(*_subslist*)

$$254\ \mathit{additions} + 1249\ \mathit{multiplications} + 1108\ \mathit{sums} + 27\ \mathit{assignments}$$

map(*cost*, eval(*soln*, *_subslist*))

$$[1043\ \mathit{additions} + 4113\ \mathit{multiplications} + 3968\ \mathit{sums} + \mathit{divisions} + \mathit{assignments},$$
$$648\ \mathit{additions} + 2631\ \mathit{multiplications} + 2487\ \mathit{sums} + \mathit{divisions} + \mathit{assignments},$$
$$648\ \mathit{additions} + 2631\ \mathit{multiplications} + 2487\ \mathit{sums} + \mathit{divisions} + \mathit{assignments},$$
$$471\ \mathit{additions} + 1988\ \mathit{multiplications} + 1842\ \mathit{sums} + \mathit{divisions} + \mathit{assignments},$$
$$648\ \mathit{additions} + 2631\ \mathit{multiplications} + 2487\ \mathit{sums} + \mathit{divisions} + \mathit{assignments}]$$

However, it optimizes well:

optsoln := optimize(*soln*, *tryhard*)

cost(*optsoln*)

$$739\ \mathit{additions} + 1061\ \mathit{multiplications} + \mathit{divisions} + 26\ \mathit{subscripts} + 447\ \mathit{assignments}$$

The following command produces an optimized procedure, which we will not show here.

makeproc(eval([*optsoln*], *_subslist*), *parameters* = [*N*, *Y*, *t*], *globals* = [*A*, *B*, *C*, D, *E*])