Calculation of an Integral for FAME

(Integral encountered in the memo by Bob Reasenberg, "Bias in the Estimate of Star Coordinates due to Spatial Variation in Detector Sensitivity", TM97-01)

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1. A Calculation of the Integral I_ν (eq. 8) in Memo TM97-01.

Start with a clean slate.

restart

1.1. Assumptions and Definitions.

Make some reasonable assumptions on the variables we'll be using, in order to help out the integrator as much as possible. We'll use g instead of γ, since γ in Maple is a reserved numerical value.

assume($0 < S$, $0 \le g$, $g < 1$, p, *real*, $0 < \lambda$, $0 < k$, ϕ, *real*, f, *real*)

about(S, g, p, λ, k, ϕ, f)

```
Originally S, renamed S~:
  is assumed to be: RealRange(Open(0),infinity)

Originally g, renamed g~:
  is assumed to be: RealRange(0,Open(1))

Originally p, renamed p~:
  is assumed to be: real

Originally lambda, renamed lambda~:
  is assumed to be: RealRange(Open(0),infinity)

Originally k, renamed k~:
  is assumed to be: RealRange(Open(0),infinity)
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Originally phi, renamed phi~:
  is assumed to be: real

Originally f, renamed f~:
  is assumed to be: real
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For part of the integrand, we need to differentiate the following expression.

$$Js := \frac{\sin\left(\frac{\pi\, p\, S}{\lambda}\right) - \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)}{\frac{\pi\, p}{\lambda}}$$

$$dJs := \frac{\partial}{\partial p} Js$$

$$dJs := \frac{\left(\frac{\cos\left(\frac{\pi\, p\, S}{\lambda}\right)\pi\, S}{\lambda} - \frac{\cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\pi\, S\, g}{\lambda}\right)\lambda}{\pi\, p} - \frac{\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right) - \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}$$

$$dJs := \mathrm{termfunc}(dJs, factor)$$

$$dJs := -\frac{S\left(-\cos\left(\frac{\pi\, p\, S}{\lambda}\right) + \cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right) g\right)}{p} + \frac{\left(-\sin\left(\frac{\pi\, p\, S}{\lambda}\right) + \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}$$

The integrand is

$$integrand := Js\, dJs \sin(k\, p + \phi)$$

$$integrand := \left(\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right) - \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda \left(-\frac{S\left(-\cos\left(\frac{\pi\, p\, S}{\lambda}\right) + \cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right) g\right)}{p} + \frac{\left(-\sin\left(\frac{\pi\, p\, S}{\lambda}\right) + \sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}\right) \sin(k\, p + \phi)\right) / (\pi\, p)$$

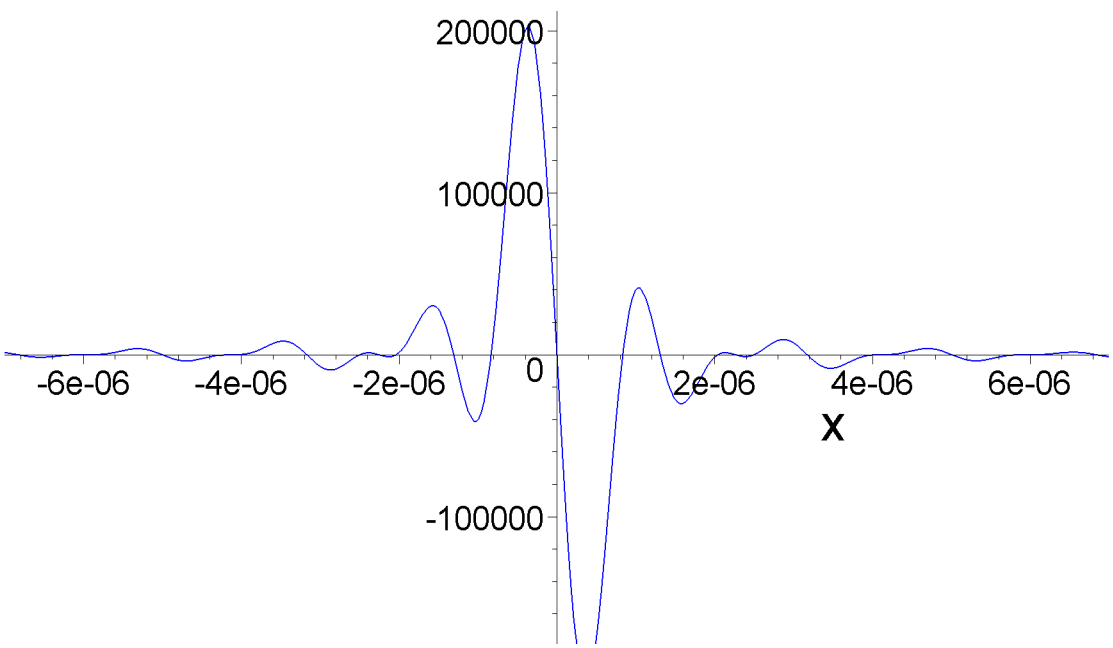
Let's get some idea what this looks like.

$$F := \mathrm{fn}(integrand, g, \phi, \lambda, k, S, p)$$

$$F := (g, \phi, \lambda, k, S, p) \rightarrow \left(\left(\sin\left(\frac{\pi\, p\sim\, S\sim}{\lambda\sim}\right) - \sin\left(\frac{\pi\, p\sim\, S\sim\, g\sim}{\lambda\sim}\right)\right)\lambda\sim \left(-\frac{S\sim\left(-\cos\left(\frac{\pi\, p\sim\, S\sim}{\lambda\sim}\right) + \cos\left(\frac{\pi\, p\sim\, S\sim\, g\sim}{\lambda\sim}\right) g\sim\right)}{p\sim} + \frac{\left(-\sin\left(\frac{\pi\, p\sim\, S\sim}{\lambda\sim}\right) + \sin\left(\frac{\pi\, p\sim\, S\sim\, g\sim}{\lambda\sim}\right)\right)\lambda\sim}{\pi\, p\sim^2}\right)\sin(k\sim\, p\sim + \phi\sim)\right) / (\pi\, p\sim)$$

$dp := .7\ 10^{-5}$

$\mathrm{plot}\left(F\left(.1, \frac{\pi}{4}, 5500\ 10^{(-10)}, .1, .6, x\right), x = -dp\ ..\ dp, color = blue, numpoints = 100, axes = normal, thickness = 2\right)$



Hmm, this appears to be an odd function, which ought to integrate to zero for integration limits symmetric about $p = 0$. Let's expand the term $\sin(k\, p + \phi)$ and consider the $\sin(k\, p)$ and $\cos(k\, p)$ terms separately.

$loc := \mathrm{location}(integrand, \sin(k\, p + \phi))$

$\mathrm{subsop}(loc = \mathrm{expand}(\mathrm{op}(loc, integrand)), integrand)$

$$\left(\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right)-\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda\left(-\frac{S\left(-\cos\left(\frac{\pi\, p\, S}{\lambda}\right)+\cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right)g\right)}{p}+\frac{\left(-\sin\left(\frac{\pi\, p\, S}{\lambda}\right)+\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}\right)(\sin(k\, p)\cos(\phi)+\cos(k\, p)\sin(\phi))\right)/(\pi\, p)$$

integrand_s := coeff(%, sin(*k p*), 1) sin(*k p*)

integrand_c := coeff(%%, cos(*k p*), 1) cos(*k p*)

$$integrand_s := \left(\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right)-\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda\left(-\frac{S\left(-\cos\left(\frac{\pi\, p\, S}{\lambda}\right)+\cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right)g\right)}{p}+\frac{\left(-\sin\left(\frac{\pi\, p\, S}{\lambda}\right)+\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}\right)\cos(\phi)\sin(k\, p)\right)/(\pi\, p)$$

$$integrand_c := \left(\left(\sin\left(\frac{\pi\, p\, S}{\lambda}\right)-\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda\left(-\frac{S\left(-\cos\left(\frac{\pi\, p\, S}{\lambda}\right)+\cos\left(\frac{\pi\, p\, S\, g}{\lambda}\right)g\right)}{p}+\frac{\left(-\sin\left(\frac{\pi\, p\, S}{\lambda}\right)+\sin\left(\frac{\pi\, p\, S\, g}{\lambda}\right)\right)\lambda}{\pi\, p^2}\right)\sin(\phi)\cos(k\, p)\right)/(\pi\, p)$$

Guard against typos...

expand(simplify(*integrand_s* + *integrand_c* − *integrand*))

0

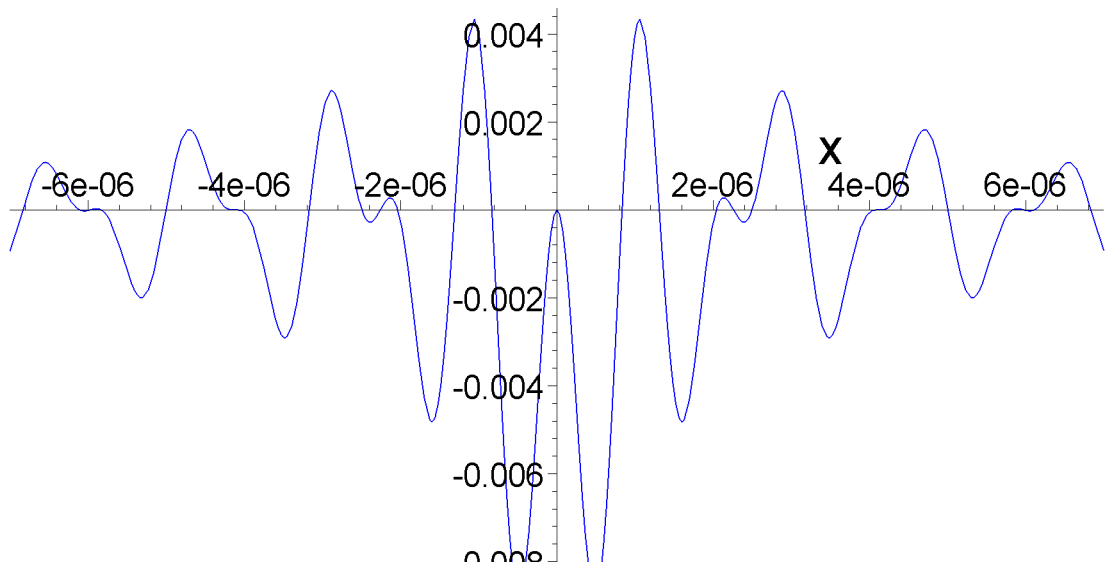
Make the two integrand contributions into functions so we can plot them.

$Fs := \mathrm{fn}(integrand_s, g, \phi, \lambda, k, S, p)$

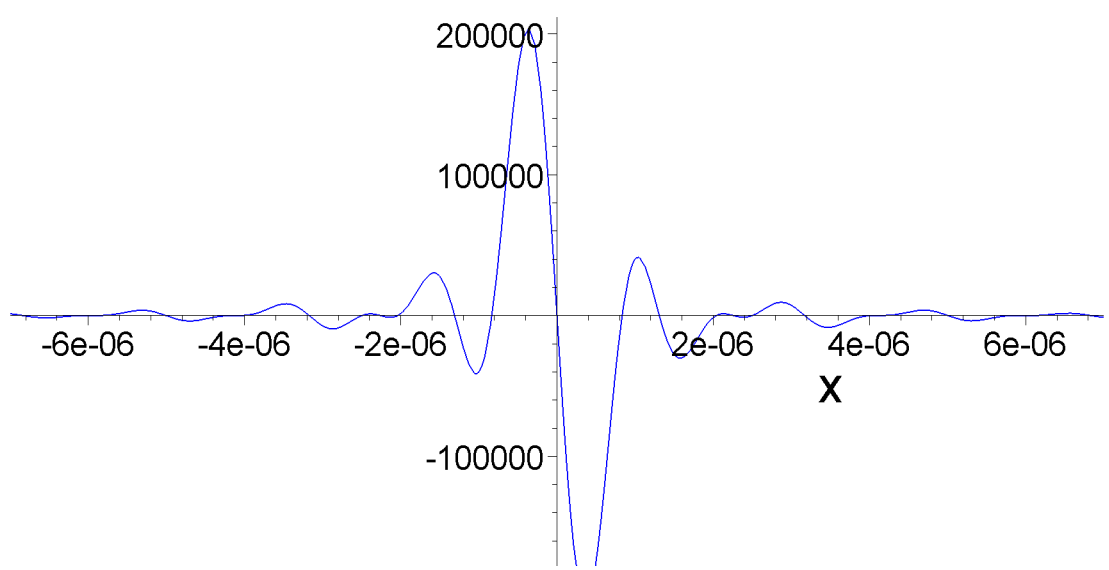
$Fc := \mathrm{fn}(integrand_c, g, \phi, \lambda, k, S, p)$

The argument $k\,p$ is mighty small for the numerical values we have to use in order to see the "wiggles", so the $\sin(k\,p)$ term ought to be miniscule compared to the $\cos(k\,p)$ term.

$$\mathrm{plot}\left(\mathrm{Fs}\left(.1, \frac{\pi}{4}, 5500\ 10^{(-10)}, .1, .6, x\right), x = -dp\ ..\ dp, axes = normal, numpoints = 100, thickness = 2\right)$$



$$\mathrm{plot}\left(\mathrm{Fc}\left(.1, \frac{\pi}{4}, 5500\ 10^{(-10)}, .1, .6, x\right), x = -dp\ ..\ dp, axes = normal, numpoints = 100, thickness = 2\right)$$



However, notice that the $\sin(k\,p)$ part of the integrand is even, while the $\cos(k\,p)$ part of the integrand is odd. Hence, for integration limits symmetric about zero, the $\cos(k\,p)$ contribution to the integral, despite the overwhelming magnitude of the $\cos(k\,p)$ term, will vanish.

1.2. Perform the Definite Integration.

$$Iv := \int_{-\infty}^{\infty} integrand\ dp$$

$$\begin{aligned}
Iv := &-\frac{1}{8} k\,\lambda\,(-\lambda\,\pi\,k\,\mathrm{signum}(2\,\pi\,S - k\,\lambda)\cos(\phi) \\
&- 2\,\lambda\,\pi\,k\,\mathrm{signum}(\pi\,S\,g - k\,\lambda - \pi\,S)\cos(\phi) - 2\,\lambda\,\pi\,k\sin(\phi)\,\mathrm{signum}(\pi\,S\,g + k\,\lambda - \pi\,S) \\
&- \lambda\,\pi\,k\sin(\phi)\,\mathrm{signum}(2\,\pi\,S - k\,\lambda) - 2\,\lambda\,\pi\,k\sin(\phi)\,\mathrm{signum}(\pi\,S\,g - k\,\lambda - \pi\,S) \\
&+ 2\sin(\phi)\,S\,g\ln(-\pi\,S\,g + k\,\lambda - \pi\,S)\,\pi + 2\sin(\phi)\,S\,g\ln(-\pi\,S\,g - k\,\lambda + \pi\,S)\,\pi \\
&+ 2\sin(\phi)\,S\,g\ln(\pi\,S\,g - k\,\lambda - \pi\,S)\,\pi - 2\sin(\phi)\,S\,g\ln(\pi\,S\,g - k\,\lambda + \pi\,S)\,\pi \\
&- 2\sin(\phi)\,S\,g\ln(\pi\,S\,g + k\,\lambda - \pi\,S)\,\pi + 2\,\pi^2\sin(\phi)\,S\,g\,\mathrm{signum}(2\,\pi\,S\,g - k\,\lambda) \\
&+ 2\,\pi^2\sin(\phi)\,S\,g\,\mathrm{signum}(\pi\,S\,g - k\,\lambda - \pi\,S) - 2\,\pi^2\sin(\phi)\,S\,g\,\mathrm{signum}(\pi\,S\,g + k\,\lambda - \pi\,S) \\
&- 2\,\pi^2\,\mathrm{signum}(\pi\,S\,g - k\,\lambda + \pi\,S)\cos(\phi)\,S\,g \\
&- 2\,\pi^2\sin(\phi)\,S\,g\,\mathrm{signum}(\pi\,S\,g - k\,\lambda + \pi\,S) + 2\,\pi^2\,\mathrm{signum}(2\,\pi\,S\,g - k\,\lambda)\cos(\phi)\,S\,g \\
&+ 2\,\pi^2\,\mathrm{signum}(\pi\,S\,g + k\,\lambda - \pi\,S)\cos(\phi)\,S\,g \\
&+ 2\,\pi^2\,\mathrm{signum}(\pi\,S\,g - k\,\lambda - \pi\,S)\cos(\phi)\,S\,g - 2\sin(\phi)\,S\,g\ln(-2\,\pi\,S\,g + k\,\lambda)\,\pi \\
&+ 2\,\lambda\,\pi\,k\sin(\phi)\,\mathrm{signum}(\pi\,S\,g - k\,\lambda + \pi\,S) + 2\,\lambda\,\pi\,k\,\mathrm{signum}(\pi\,S\,g - k\,\lambda + \pi\,S)\cos(\phi) \\
&+ 2\,\lambda\,\pi\,k\,\mathrm{signum}(\pi\,S\,g + k\,\lambda - \pi\,S)\cos(\phi) - \lambda\,\pi\,k\,\mathrm{signum}(2\,\pi\,S\,g - k\,\lambda)\cos(\phi) \\
&- \lambda\,\pi\,k\sin(\phi)\,\mathrm{signum}(2\,\pi\,S\,g - k\,\lambda) - 2\sin(\phi)\,\lambda\,k\ln(\pi\,S\,g + k\,\lambda - \pi\,S) \\
&- \sin(\phi)\,\lambda\,k\ln(2\,\pi\,S - k\,\lambda) - \sin(\phi)\,\lambda\,k\ln(2\,\pi\,S\,g - k\,\lambda) \\
&+ 2\sin(\phi)\,\lambda\,k\ln(\pi\,S\,g - k\,\lambda + \pi\,S) - 2\sin(\phi)\,\lambda\,k\ln(\pi\,S\,g - k\,\lambda - \pi\,S) \\
&- 2\sin(\phi)\,\lambda\,k\ln(-\pi\,S\,g + k\,\lambda - \pi\,S) + \sin(\phi)\,\lambda\,k\ln(-2\,\pi\,S + k\,\lambda) \\
&+ 2\sin(\phi)\,\lambda\,k\ln(-\pi\,S\,g + k\,\lambda + \pi\,S) + 2\sin(\phi)\,\lambda\,k\ln(-\pi\,S\,g - k\,\lambda + \pi\,S) \\
&+ \sin(\phi)\,\lambda\,k\ln(-2\,\pi\,S\,g + k\,\lambda) - 2\,\pi^2\sin(\phi)\,S\,\mathrm{signum}(\pi\,S\,g - k\,\lambda + \pi\,S) \\
&- 2\,\pi^2\sin(\phi)\,S\,\mathrm{signum}(\pi\,S\,g - k\,\lambda - \pi\,S) - 2\sin(\phi)\,S\ln(-\pi\,S\,g - k\,\lambda + \pi\,S)\,\pi \\
&+ 2\sin(\phi)\,S\ln(\pi\,S\,g + k\,\lambda - \pi\,S)\,\pi - 2\sin(\phi)\,S\ln(-2\,\pi\,S + k\,\lambda)\,\pi \\
&+ 2\sin(\phi)\,S\ln(-\pi\,S\,g + k\,\lambda - \pi\,S)\,\pi + 2\sin(\phi)\,S\ln(-\pi\,S\,g + k\,\lambda + \pi\,S)\,\pi \\
&- 2\sin(\phi)\,S\ln(\pi\,S\,g - k\,\lambda - \pi\,S)\,\pi + 2\sin(\phi)\,S\ln(2\,\pi\,S - k\,\lambda)\,\pi \\
&- 2\sin(\phi)\,S\ln(\pi\,S\,g - k\,\lambda + \pi\,S)\,\pi + 2\,\pi^2\sin(\phi)\,S\,\mathrm{signum}(\pi\,S\,g + k\,\lambda - \pi\,S)
\end{aligned}$$

$$+ 2\pi^2 \operatorname{signum}(2\pi S - k\lambda)\cos(\phi) S - 2\pi^2 \operatorname{signum}(\pi S g + k\lambda - \pi S)\cos(\phi) S$$
$$+ 2\pi^2 \sin(\phi) S \operatorname{signum}(2\pi S - k\lambda) - 2\pi^2 \operatorname{signum}(\pi S g - k\lambda - \pi S)\cos(\phi) S$$
$$- 2\pi^2 \operatorname{signum}(\pi S g - k\lambda + \pi S)\cos(\phi) S - 4\lambda\pi k\cos(\phi)$$
$$+ 2\sin(\phi) S g \ln(2\pi S g - k\lambda)\pi - 2\sin(\phi) S g \ln(-\pi S g + k\lambda + \pi S)\pi \Big/ \pi^2$$

cost(%)

450 *multiplications* + 140 *additions* + 105 *functions* + *divisions*

1.3. Simplification of the Result.

Well, that's a mess. Let's see how much we can clean it up.

Iv := collect(*Iv*, [sin(ϕ), cos(ϕ), signum, ln], *factor*)

$$Iv := \Bigg(-\frac{1}{8}\frac{k\lambda(2\pi S g - k\lambda)\operatorname{signum}(2\pi S g - k\lambda)}{\pi}$$
$$+\frac{1}{4}\frac{k\lambda(\pi S g + k\lambda - \pi S)\operatorname{signum}(\pi S g + k\lambda - \pi S)}{\pi}$$
$$+\frac{1}{4}\frac{k\lambda(\pi S g - k\lambda + \pi S)\operatorname{signum}(\pi S g - k\lambda + \pi S)}{\pi}$$
$$-\frac{1}{8}\frac{k\lambda(2\pi S - k\lambda)\operatorname{signum}(2\pi S - k\lambda)}{\pi}$$
$$-\frac{1}{4}\frac{k\lambda(\pi S g - k\lambda - \pi S)\operatorname{signum}(\pi S g - k\lambda - \pi S)}{\pi}$$
$$-\frac{1}{8}\frac{k\lambda(2\pi S g - k\lambda)\ln(2\pi S g - k\lambda)}{\pi^2}$$
$$+\frac{1}{4}\frac{k\lambda(\pi S g - k\lambda + \pi S)\ln(\pi S g - k\lambda + \pi S)}{\pi^2}$$
$$+\frac{1}{8}\frac{k\lambda(2\pi S g - k\lambda)\ln(-2\pi S g + k\lambda)}{\pi^2} + \frac{1}{8}\frac{k\lambda(2\pi S - k\lambda)\ln(-2\pi S + k\lambda)}{\pi^2}$$
$$-\frac{1}{4}\frac{k\lambda(\pi S g - k\lambda + \pi S)\ln(-\pi S g + k\lambda - \pi S)}{\pi^2}$$
$$-\frac{1}{4}\frac{k\lambda(\pi S g + k\lambda - \pi S)\ln(-\pi S g - k\lambda + \pi S)}{\pi^2}$$
$$+\frac{1}{4}\frac{k\lambda(\pi S g - k\lambda - \pi S)\ln(-\pi S g + k\lambda + \pi S)}{\pi^2} - \frac{1}{8}\frac{k\lambda(2\pi S - k\lambda)\ln(2\pi S - k\lambda)}{\pi^2}$$

$$+\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g+k\,\lambda-\pi\,S)\,\ln(\pi\,S\,g+k\,\lambda-\pi\,S)}{\pi^2}$$

$$-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda-\pi\,S)\,\ln(\pi\,S\,g-k\,\lambda-\pi\,S)}{\pi^2}\Bigg)\sin(\phi)+\Bigg($$

$$-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S\,g-k\,\lambda)\,\mathrm{signum}(2\,\pi\,S\,g-k\,\lambda)}{\pi}$$

$$-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g+k\,\lambda-\pi\,S)\,\mathrm{signum}(\pi\,S\,g+k\,\lambda-\pi\,S)}{\pi}$$

$$+\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda+\pi\,S)\,\mathrm{signum}(\pi\,S\,g-k\,\lambda+\pi\,S)}{\pi}$$

$$-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S-k\,\lambda)\,\mathrm{signum}(2\,\pi\,S-k\,\lambda)}{\pi}$$

$$-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda-\pi\,S)\,\mathrm{signum}(\pi\,S\,g-k\,\lambda-\pi\,S)}{\pi}+\frac{1}{2}\frac{k^2\,\lambda^2}{\pi}\Bigg)\cos(\phi)$$

cost(%)

84 *additions* + 248 *multiplications* + 22 *functions* + 21 *divisions*

Iv := convert(*Iv*, abs)

$$Iv := \Bigg(-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S\,g-k\,\lambda)^2}{\pi\,|2\,\pi\,S\,g-k\,\lambda|}+\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g+k\,\lambda-\pi\,S)^2}{\pi\,|\pi\,S\,g+k\,\lambda-\pi\,S|}+\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda+\pi\,S)^2}{\pi\,|\pi\,S\,g-k\,\lambda+\pi\,S|}$$

$$-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S-k\,\lambda)^2}{\pi\,|2\,\pi\,S-k\,\lambda|}-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda-\pi\,S)^2}{\pi\,|\pi\,S\,g-k\,\lambda-\pi\,S|}$$

$$-\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S\,g-k\,\lambda)\,\ln(2\,\pi\,S\,g-k\,\lambda)}{\pi^2}$$

$$+\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda+\pi\,S)\,\ln(\pi\,S\,g-k\,\lambda+\pi\,S)}{\pi^2}$$

$$+\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S\,g-k\,\lambda)\,\ln(-2\,\pi\,S\,g+k\,\lambda)}{\pi^2}+\frac{1}{8}\frac{k\,\lambda\,(2\,\pi\,S-k\,\lambda)\,\ln(-2\,\pi\,S+k\,\lambda)}{\pi^2}$$

$$-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g-k\,\lambda+\pi\,S)\,\ln(-\pi\,S\,g+k\,\lambda-\pi\,S)}{\pi^2}$$

$$-\frac{1}{4}\frac{k\,\lambda\,(\pi\,S\,g+k\,\lambda-\pi\,S)\,\ln(-\pi\,S\,g-k\,\lambda+\pi\,S)}{\pi^2}$$

$$+\frac{1}{4}\frac{k\lambda(\pi S g-k\lambda-\pi S)\ln(-\pi S g+k\lambda+\pi S)}{\pi^2}-\frac{1}{8}\frac{k\lambda(2\pi S-k\lambda)\ln(2\pi S-k\lambda)}{\pi^2}$$

$$+\frac{1}{4}\frac{k\lambda(\pi S g+k\lambda-\pi S)\ln(\pi S g+k\lambda-\pi S)}{\pi^2}$$

$$-\frac{1}{4}\frac{k\lambda(\pi S g-k\lambda-\pi S)\ln(\pi S g-k\lambda-\pi S)}{\pi^2}\Bigg)\sin(\phi)+\Bigg(-\frac{1}{8}\frac{k\lambda(2\pi S g-k\lambda)^2}{\pi\left|2\pi S g-k\lambda\right|}$$

$$-\frac{1}{4}\frac{k\lambda(\pi S g+k\lambda-\pi S)^2}{\pi\left|\pi S g+k\lambda-\pi S\right|}+\frac{1}{4}\frac{k\lambda(\pi S g-k\lambda+\pi S)^2}{\pi\left|\pi S g-k\lambda+\pi S\right|}-\frac{1}{8}\frac{k\lambda(2\pi S-k\lambda)^2}{\pi\left|2\pi S-k\lambda\right|}$$

$$-\frac{1}{4}\frac{k\lambda(\pi S g-k\lambda-\pi S)^2}{\pi\left|\pi S g-k\lambda-\pi S\right|}+\frac{1}{2}\frac{k^2\lambda^2}{\pi}\Bigg)\cos(\phi)$$

Now let's make the substitution $f=\frac{k\lambda}{\pi S}$.

collect$\left(\text{subs}\left(\lambda=\frac{f\pi S}{k}, Iv\right), [\sin, \cos, \ln]\right)$

$$\Bigg(-\frac{1}{8}\frac{f S(2\pi S g-f\pi S)^2}{\left|2\pi S g-f\pi S\right|}+\frac{1}{4}\frac{f S(\pi S g+f\pi S-\pi S)^2}{\left|\pi S g+f\pi S-\pi S\right|}+\frac{1}{4}\frac{f S(\pi S g-f\pi S+\pi S)^2}{\left|\pi S g-f\pi S+\pi S\right|}$$

$$-\frac{1}{8}\frac{f S(2\pi S-f\pi S)^2}{\left|2\pi S-f\pi S\right|}-\frac{1}{4}\frac{f S(\pi S g-f\pi S-\pi S)^2}{\left|\pi S g-f\pi S-\pi S\right|}$$

$$-\frac{1}{8}\frac{f S(2\pi S g-f\pi S)\ln(2\pi S g-f\pi S)}{\pi}$$

$$+\frac{1}{4}\frac{f S(\pi S g-f\pi S+\pi S)\ln(\pi S g-f\pi S+\pi S)}{\pi}$$

$$+\frac{1}{8}\frac{f S(2\pi S g-f\pi S)\ln(-2\pi S g+f\pi S)}{\pi}+\frac{1}{8}\frac{f S(2\pi S-f\pi S)\ln(-2\pi S+f\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{f S(\pi S g-f\pi S+\pi S)\ln(-\pi S g+f\pi S-\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{f S(\pi S g+f\pi S-\pi S)\ln(-\pi S g-f\pi S+\pi S)}{\pi}$$

$$+\frac{1}{4}\frac{f S(\pi S g-f\pi S-\pi S)\ln(-\pi S g+f\pi S+\pi S)}{\pi}$$

$$-\frac{1}{8}\frac{f S(2\pi S-f\pi S)\ln(2\pi S-f\pi S)}{\pi}$$

$$+\frac{1}{4}\frac{f S(\pi S g+f\pi S-\pi S)\ln(\pi S g+f\pi S-\pi S)}{\pi}$$

$$-\frac{1}{4}\frac{f S (\pi S g - f \pi S - \pi S) \ln(\pi S g - f \pi S - \pi S)}{\pi}\Bigg)\sin(\phi) + \Bigg(-\frac{1}{8}\frac{f S (2 \pi S g - f \pi S)^2}{|2 \pi S g - f \pi S|}$$

$$-\frac{1}{4}\frac{f S (\pi S g + f \pi S - \pi S)^2}{|\pi S g + f \pi S - \pi S|} + \frac{1}{4}\frac{f S (\pi S g - f \pi S + \pi S)^2}{|\pi S g - f \pi S + \pi S|} - \frac{1}{8}\frac{f S (2 \pi S - f \pi S)^2}{|2 \pi S - f \pi S|}$$

$$-\frac{1}{4}\frac{f S (\pi S g - f \pi S - \pi S)^2}{|\pi S g - f \pi S - \pi S|} + \frac{1}{2} f^2 \pi S^2\Bigg)\cos(\phi)$$

$L_1 :=$ location(%, coeff(%, sin(ϕ), 1))

$$L_1 := [1, 1]$$

$L_2 :=$ location(%%, coeff(%%, cos(ϕ), 1))

$$L_2 := [2, 1]$$

subsop($L_1 =$ map($x \to$ simplify(factor(x)), coeff(%%%, sin(ϕ), 1)),
$L_2 =$ map($x \to$ simplify(factor(x)), coeff(%%%, cos(ϕ), 1)), %%%)

$$\Bigg(-\frac{1}{8} f \pi S^2 |2 g - f| + \frac{1}{4} f \pi S^2 |g + f - 1| + \frac{1}{4} f \pi |g - f + 1| S^2 - \frac{1}{8} f \pi |-2 + f| S^2$$

$$-\frac{1}{4}|g - f - 1| f \pi S^2 - \frac{1}{8} f S^2 (2 g - f)(\ln(\pi) + \ln(S) + \ln(2 g - f))$$

$$+\frac{1}{4} f S^2 (g - f + 1)(\ln(\pi) + \ln(S) + \ln(g - f + 1))$$

$$+\frac{1}{8} f S^2 (2 g - f)(\ln(\pi) + \ln(S) + \ln(-2 g + f))$$

$$-\frac{1}{8} f S^2 (-2 + f)(\ln(\pi) + \ln(S) + \ln(-2 + f))$$

$$-\frac{1}{4} f S^2 (g - f + 1)(\ln(\pi) + \ln(S) + \ln(-g + f - 1))$$

$$-\frac{1}{4} f S^2 (g + f - 1)(\ln(\pi) + \ln(S) + \ln(-g - f + 1))$$

$$+\frac{1}{4} f S^2 (g - f - 1)(\ln(\pi) + \ln(S) + \ln(-g + f + 1))$$

$$+\frac{1}{8} f S^2 (-2 + f)(\ln(\pi) + \ln(S) + \ln(2 - f))$$

$$+\frac{1}{4} f S^2 (g + f - 1)(\ln(\pi) + \ln(S) + \ln(g + f - 1))$$

$$-\frac{1}{4} f S^2 (g-f-1)(\ln(\pi)+\ln(S)+\ln(g-f-1))\Big)\sin(\phi)+\Big(-\frac{1}{8} f \pi S^2 |2g-f|$$
$$-\frac{1}{4} f \pi S^2 |g+f-1|+\frac{1}{4} f \pi |g-f+1| S^2-\frac{1}{8} f \pi |-2+f| S^2-\frac{1}{4}|g-f-1| f \pi S^2$$
$$+\frac{1}{2} f^2 \pi S^2\Big)\cos(\phi)$$

$$\frac{\pi f S^2 \operatorname{subs}\left(Q=\pi, \operatorname{collect}\left(\operatorname{coeff}\left(\operatorname{subs}\left(\pi=Q, \operatorname{factor}\left(\frac{8\,\%}{\pi f}\right)\right), S, 2\right), [\sin, \cos, Q, \ln], factor\right)\right)}{8}$$

$$\frac{1}{8}\pi f S^2 ((-2|g-f-1|+2|g+f-1|-|2g-f|+2|g-f+1|-|-2+f|+($$
$$(2f-2-2g)\ln(-g+f-1)+(-2f-2g+2)\ln(-g-f+1)+(2g-f)\ln(-2g+f)$$
$$+(-2f-2+2g)\ln(-g+f+1)+(2-f)\ln(-2+f)+(-2+f)\ln(2-f)$$
$$+(2g+2f-2)\ln(g+f-1)+(-2g+2f+2)\ln(g-f-1)+(-2g+f)\ln(2g-f)$$
$$+(2+2g-2f)\ln(g-f+1))/\pi)\sin(\phi)$$
$$+(2|g-f+1|+4f-|2g-f|-2|g+f-1|-2|g-f-1|-|-2+f|)\cos(\phi))$$

Check:

$$\operatorname{simplify}\left(\operatorname{subs}\left(f=\frac{k\lambda}{\pi S}, \%\right)-Iv\right)$$

$$0$$

$Iv := \%\%$

$\operatorname{cost}(Iv)$

$$42\ multiplications + 68\ additions + 22\ functions + divisions$$

Much better. Notice there are both $\sin\phi$ and $\cos\phi$ terms, and that the latter is much simpler in form.

$Iv_sin := \operatorname{coeff}(Iv, \sin(\phi), 1)\sin(\phi)$

$$Iv_sin := \frac{1}{8}\pi f S^2 (-2|g-f-1|+2|g+f-1|-|2g-f|+2|g-f+1|-|-2+f|+($$
$$(2f-2-2g)\ln(-g+f-1)+(-2f-2g+2)\ln(-g-f+1)+(2g-f)\ln(-2g+f)$$
$$+(-2f-2+2g)\ln(-g+f+1)+(2-f)\ln(-2+f)+(-2+f)\ln(2-f)$$
$$+(2g+2f-2)\ln(g+f-1)+(-2g+2f+2)\ln(g-f-1)+(-2g+f)\ln(2g-f)$$
$$+(2+2g-2f)\ln(g-f+1))/\pi)\sin(\phi)$$

$Iv_cos := \operatorname{coeff}(Iv, \cos(\phi), 1)\cos(\phi)$

$Iv_cos :=$

$$\frac{1}{8} f S^2 \pi \left(2\left|g-f+1\right|+4f-\left|2g-f\right|-2\left|g+f-1\right|-2\left|g-f-1\right|-\left|-2+f\right|\right)\cos(\phi)$$

1.4. Characterization of the *sin* and *cos* Terms.

Let's make the coefficients of $\frac{\pi S^2 \sin(\phi)}{8}$ and $\frac{\pi S^2 \cos(\phi)}{8}$ into functions of (*g*,*f*).

$$G_s := \mathrm{fn}\left(\mathrm{coeff}\left(\mathrm{algsubs}\left(\frac{\pi \sin(\phi) S^2}{8} = Q, Iv_sin\right), Q, 1\right), g, f\right)$$

$$G_c := \mathrm{fn}\left(\mathrm{coeff}\left(\mathrm{algsubs}\left(\frac{\pi \cos(\phi) S^2}{8} = Q, Iv_cos\right), Q, 1\right), g, f\right)$$

Notice that the collection of *ln* terms in the *sin* term make for a very difficult time of keeping the *sin* term purely real. In fact, setting $g = 0$ we have $G_s(0, f)$

$$\begin{aligned} & f\left(-2\left|f+1\right|+4\left|f-1\right|-\left|f\right|-\left|-2+f\right|+\left(\ln(2-f)f-\ln(-2+f)f+\ln(-f)f\right.\right. \\ & \quad -4\ln(1-f)f+4\ln(f-1)f-\ln(f)f-2\ln(f+1)f+2\ln(-2+f)+2\ln(-1-f)f \\ & \quad \left.\left.-2\ln(2-f)+2\ln(-1-f)+4\ln(1-f)-2\ln(f+1)-4\ln(f-1)\right)/\pi\right) \end{aligned}$$

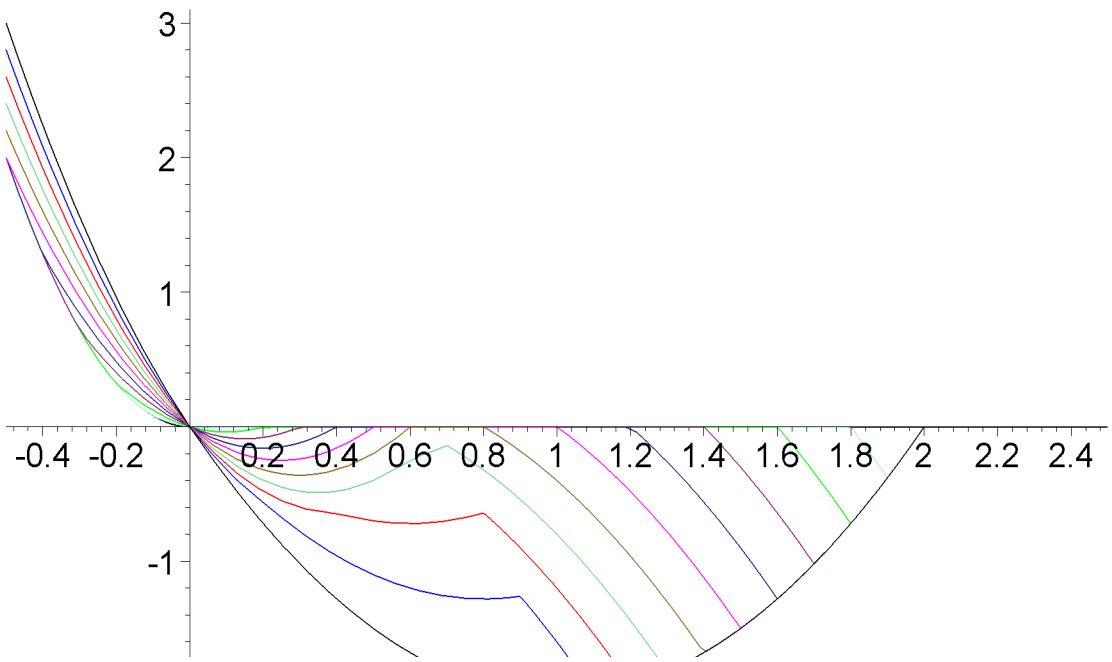
where it becomes quite clear we're in trouble. There is no real value of *f* for which this expression does not have an imaginary component. On the other hand, the *cos* term is comparatively well-behaved: $G_c(g, f)$

$$f\left(2\left|g-f+1\right|+4f-\left|2g-f\right|-2\left|g+f-1\right|-2\left|g-f-1\right|-\left|-2+f\right|\right)$$

Let's take a look at the *cos* term for various values of *g*.

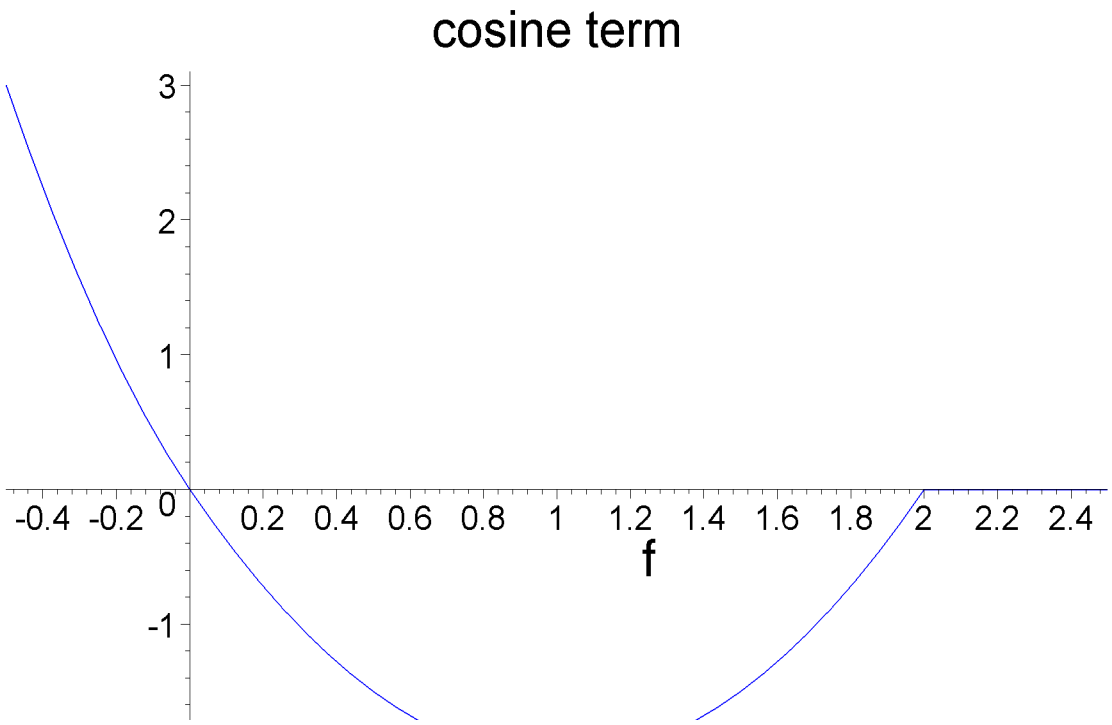
```
cosplot := proc( gvals::list, frange::range )
  local p, k;
  p := [];
  for k from 1 to nops(gvals) do
    p := [ op(p), plot( G[c](gvals[k],f), f=frange,
                        color=mycolors[(k-1 mod 10)+1],
                        thickness=2 ) ];
  od;
  plots[display]( p, axes=normal );
end:
```

$$\mathrm{cosplot}([\mathrm{seq}(.1\,i, i = 0 .. 10)], -.5 .. 2.5)$$



An animation of this is also instructive.

$plots_{animate}(G_c(g, f), f = -.5 .. 2.5, g = 0 .. 1, color = blue, axes = normal, thickness = 2, labels = [\text{"f"}, \text{""}], title = \text{"cosine term"})$



Hmm, disturbing that it is asymmetric.

2. Integration Check.

Since both Mathcad 5.0 and Mathcad 6.0 were unable to do the I_v integral and therefore provide an independent check, we should differentiate the integral and make sure we recover the integrand. First, do the indefinite integral.

$$Iv_indef := \int integrand\ dp$$

$$Iv_indef := \lambda \left(\frac{1}{4} S (2\pi S - k\lambda) \left(-\frac{\cos\left(\frac{(2\pi S - k\lambda)p}{\lambda} - \phi\right)\lambda}{(2\pi S - k\lambda)p} - \mathrm{Si}\left(\frac{(2\pi S - k\lambda)p}{\lambda}\right)\cos(\phi) + \mathrm{Ci}\left(\frac{(2\pi S - k\lambda)p}{\lambda}\right)\sin(\phi) \right) / \lambda - \frac{1}{4} S (1+g)(\pi S g - k\lambda + \pi S) \left(-\frac{\cos\left(\frac{(\pi S g - k\lambda + \pi S)p}{\lambda} - \phi\right)\lambda}{(\pi S g - k\lambda + \pi S)p} - \mathrm{Si}\left(\frac{(\pi S g - k\lambda + \pi S)p}{\lambda}\right)\cos(\phi) + \mathrm{Ci}\left(\frac{(\pi S g - k\lambda + \pi S)p}{\lambda}\right)\sin(\phi) \right) / \lambda - \frac{1}{4} S (2\pi S + k\lambda) \left(-\frac{\cos\left(\frac{(2\pi S + k\lambda)p}{\lambda} + \phi\right)\lambda}{(2\pi S + k\lambda)p} + \mathrm{Si}\left(-\frac{(2\pi S + k\lambda)p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{(2\pi S + k\lambda)p}{\lambda}\right)\sin(\phi) \right) / \lambda - \frac{1}{4} S (g-1)(\pi S g + k\lambda - \pi S) \left(-\frac{\cos\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda - \pi S)p} + \mathrm{Si}\left(-\frac{(\pi S g + k\lambda - \pi S)p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda}\right)\sin(\phi) \right) / \lambda + \frac{1}{4} S (g-1)(\pi S g - k\lambda - \pi S) \left(-\frac{\cos\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda} - \phi\right)\lambda}{(\pi S g - k\lambda - \pi S)p} - \mathrm{Si}\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\cos(\phi) \right)$$

$$+\mathrm{Ci}\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\sin(\phi)\Bigg)/\lambda + \frac{1}{4}S(1+g)(\pi S g + k\lambda + \pi S)\Bigg($$

$$-\frac{\cos\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda + \pi S)p} + \mathrm{Si}\left(-\frac{(\pi S g + k\lambda + \pi S)p}{\lambda}\right)\cos(\phi)$$

$$-\mathrm{Ci}\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda}\right)\sin(\phi)\Bigg)/\lambda + \frac{1}{4}S g(2\pi S g - k\lambda)\Bigg($$

$$-\frac{\cos\left(\frac{(2\pi S g - k\lambda)p}{\lambda} - \phi\right)\lambda}{(2\pi S g - k\lambda)p} - \mathrm{Si}\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\cos(\phi)$$

$$+\mathrm{Ci}\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\sin(\phi)\Bigg)/\lambda - \frac{1}{4}S g(2\pi S g + k\lambda)\Bigg(-\frac{\cos\left(\frac{(2\pi S g + k\lambda)p}{\lambda} + \phi\right)\lambda}{(2\pi S g + k\lambda)p}$$

$$+\mathrm{Si}\left(-\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\cos(\phi) - \mathrm{Ci}\left(\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\sin(\phi)\Bigg)/\lambda$$

$$-\frac{\lambda k^2\left(-\frac{1}{2}\frac{\sin(k p + \phi)}{k^2 p^2} - \frac{1}{2}\frac{\cos(k p + \phi)}{k p} + \frac{1}{2}\mathrm{Si}(-k p)\cos(\phi) - \frac{1}{2}\mathrm{Ci}(k p)\sin(\phi)\right)}{\pi} + \frac{1}{4}$$

$$(2\pi S + k\lambda)^2\Bigg(-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S + k\lambda)p}{\lambda} + \phi\right)\lambda^2}{(2\pi S + k\lambda)^2 p^2} - \frac{1}{2}\frac{\cos\left(\frac{(2\pi S + k\lambda)p}{\lambda} + \phi\right)\lambda}{(2\pi S + k\lambda)p}$$

$$+\frac{1}{2}\mathrm{Si}\left(-\frac{(2\pi S + k\lambda)p}{\lambda}\right)\cos(\phi) - \frac{1}{2}\mathrm{Ci}\left(\frac{(2\pi S + k\lambda)p}{\lambda}\right)\sin(\phi)\Bigg)/(\pi\lambda) - \frac{1}{4}(2\pi S - k\lambda)^2$$

$$\Bigg(-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S - k\lambda)p}{\lambda} - \phi\right)\lambda^2}{(2\pi S - k\lambda)^2 p^2} - \frac{1}{2}\frac{\cos\left(\frac{(2\pi S - k\lambda)p}{\lambda} - \phi\right)\lambda}{(2\pi S - k\lambda)p}$$

$$-\frac{1}{2}\mathrm{Si}\left(\frac{(2\pi S - k\lambda)p}{\lambda}\right)\cos(\phi) + \frac{1}{2}\mathrm{Ci}\left(\frac{(2\pi S - k\lambda)p}{\lambda}\right)\sin(\phi)\Bigg)/(\pi\lambda) + \frac{1}{2}$$

$$(\pi S g + k\lambda - \pi S)^2 \left(-\frac{1}{2} \frac{\sin\left(\frac{(\pi S g + k\lambda - \pi S)\, p}{\lambda} + \phi\right)\lambda^2}{(\pi S g + k\lambda - \pi S)^2 p^2} \right.$$

$$-\frac{1}{2} \frac{\cos\left(\frac{(\pi S g + k\lambda - \pi S)\, p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda - \pi S)\, p} + \frac{1}{2} \mathrm{Si}\left(-\frac{(\pi S g + k\lambda - \pi S)\, p}{\lambda}\right)\cos(\phi)$$

$$\left. -\frac{1}{2} \mathrm{Ci}\left(\frac{(\pi S g + k\lambda - \pi S)\, p}{\lambda}\right)\sin(\phi) \right) / (\pi\lambda) + \frac{1}{2} (\pi S g - k\lambda + \pi S)^2 \left(\right.$$

$$-\frac{1}{2} \frac{\sin\left(\frac{(\pi S g - k\lambda + \pi S)\, p}{\lambda} - \phi\right)\lambda^2}{(\pi S g - k\lambda + \pi S)^2 p^2} - \frac{1}{2} \frac{\cos\left(\frac{(\pi S g - k\lambda + \pi S)\, p}{\lambda} - \phi\right)\lambda}{(\pi S g - k\lambda + \pi S)\, p}$$

$$\left. -\frac{1}{2} \mathrm{Si}\left(\frac{(\pi S g - k\lambda + \pi S)\, p}{\lambda}\right)\cos(\phi) + \frac{1}{2} \mathrm{Ci}\left(\frac{(\pi S g - k\lambda + \pi S)\, p}{\lambda}\right)\sin(\phi) \right) / (\pi\lambda) - \frac{1}{2}$$

$$(\pi S g + k\lambda + \pi S)^2 \left(-\frac{1}{2} \frac{\sin\left(\frac{(\pi S g + k\lambda + \pi S)\, p}{\lambda} + \phi\right)\lambda^2}{(\pi S g + k\lambda + \pi S)^2 p^2} \right.$$

$$-\frac{1}{2} \frac{\cos\left(\frac{(\pi S g + k\lambda + \pi S)\, p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda + \pi S)\, p} + \frac{1}{2} \mathrm{Si}\left(-\frac{(\pi S g + k\lambda + \pi S)\, p}{\lambda}\right)\cos(\phi)$$

$$\left. -\frac{1}{2} \mathrm{Ci}\left(\frac{(\pi S g + k\lambda + \pi S)\, p}{\lambda}\right)\sin(\phi) \right) / (\pi\lambda) - \frac{1}{2} (\pi S g - k\lambda - \pi S)^2 \left(\right.$$

$$-\frac{1}{2} \frac{\sin\left(\frac{(\pi S g - k\lambda - \pi S)\, p}{\lambda} - \phi\right)\lambda^2}{(\pi S g - k\lambda - \pi S)^2 p^2} - \frac{1}{2} \frac{\cos\left(\frac{(\pi S g - k\lambda - \pi S)\, p}{\lambda} - \phi\right)\lambda}{(\pi S g - k\lambda - \pi S)\, p}$$

$$\left. -\frac{1}{2} \mathrm{Si}\left(\frac{(\pi S g - k\lambda - \pi S)\, p}{\lambda}\right)\cos(\phi) + \frac{1}{2} \mathrm{Ci}\left(\frac{(\pi S g - k\lambda - \pi S)\, p}{\lambda}\right)\sin(\phi) \right) / (\pi\lambda) + \frac{1}{4}$$

$$(2\pi S g + k\lambda)^2 \left(-\frac{1}{2} \frac{\sin\left(\frac{(2\pi S g + k\lambda)\, p}{\lambda} + \phi\right)\lambda^2}{(2\pi S g + k\lambda)^2 p^2} - \frac{1}{2} \frac{\cos\left(\frac{(2\pi S g + k\lambda)\, p}{\lambda} + \phi\right)\lambda}{(2\pi S g + k\lambda)\, p} \right.$$

$$+\frac{1}{2}\,\mathrm{Si}\left(-\frac{(2\,\pi\,S\,g+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)-\frac{1}{2}\,\mathrm{Ci}\left(\frac{(2\,\pi\,S\,g+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)\Bigg)/(\pi\,\lambda)-\frac{1}{4}$$

$$(2\,\pi\,S\,g-k\,\lambda)^2\left(-\frac{1}{2}\,\frac{\sin\left(\frac{(2\,\pi\,S\,g-k\,\lambda)\,p}{\lambda}-\phi\right)\lambda^2}{(2\,\pi\,S\,g-k\,\lambda)^2\,p^2}-\frac{1}{2}\,\frac{\cos\left(\frac{(2\,\pi\,S\,g-k\,\lambda)\,p}{\lambda}-\phi\right)\lambda}{(2\,\pi\,S\,g-k\,\lambda)\,p}\right.$$

$$\left.-\frac{1}{2}\,\mathrm{Si}\left(\frac{(2\,\pi\,S\,g-k\,\lambda)\,p}{\lambda}\right)\cos(\phi)+\frac{1}{2}\,\mathrm{Ci}\left(\frac{(2\,\pi\,S\,g-k\,\lambda)\,p}{\lambda}\right)\sin(\phi)\right)/(\pi\,\lambda)\Bigg)/\pi$$

cost(%)

598 *multiplications* + 134 *divisions* + 233 *additions* + 94 *functions*

Now differentiate this wrt *p*.

$\frac{\partial}{\partial p}\,Iv_indef$

$$\lambda\left(\frac{1}{4}\,S\,(2\,\pi\,S-k\,\lambda)\left(\frac{\sin\left(\frac{(2\,\pi\,S-k\,\lambda)\,p}{\lambda}-\phi\right)}{p}+\frac{\cos\left(\frac{(2\,\pi\,S-k\,\lambda)\,p}{\lambda}-\phi\right)\lambda}{(2\,\pi\,S-k\,\lambda)\,p^2}\right.\right.$$

$$\left.-\frac{\sin\left(\frac{(2\,\pi\,S-k\,\lambda)\,p}{\lambda}\right)\cos(\phi)}{p}+\frac{\cos\left(\frac{(2\,\pi\,S-k\,\lambda)\,p}{\lambda}\right)\sin(\phi)}{p}\right)/\lambda-\frac{1}{4}\,S\,(1+g)$$

$$(\pi\,S\,g-k\,\lambda+\pi\,S)\left(\frac{\sin\left(\frac{(\pi\,S\,g-k\,\lambda+\pi\,S)\,p}{\lambda}-\phi\right)}{p}+\frac{\cos\left(\frac{(\pi\,S\,g-k\,\lambda+\pi\,S)\,p}{\lambda}-\phi\right)\lambda}{(\pi\,S\,g-k\,\lambda+\pi\,S)\,p^2}\right.$$

$$\left.-\frac{\sin\left(\frac{(\pi\,S\,g-k\,\lambda+\pi\,S)\,p}{\lambda}\right)\cos(\phi)}{p}+\frac{\cos\left(\frac{(\pi\,S\,g-k\,\lambda+\pi\,S)\,p}{\lambda}\right)\sin(\phi)}{p}\right)/\lambda-\frac{1}{4}\,S$$

$$(2\,\pi\,S+k\,\lambda)\left(\frac{\sin\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)}{p}+\frac{\cos\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}+\phi\right)\lambda}{(2\,\pi\,S+k\,\lambda)\,p^2}\right.$$

$$\left.\left.-\frac{\sin\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\cos(\phi)}{p}-\frac{\cos\left(\frac{(2\,\pi\,S+k\,\lambda)\,p}{\lambda}\right)\sin(\phi)}{p}\right)/\lambda-\frac{1}{4}\,S\,(g-1)\right.$$

$$(\pi S g + k\lambda - \pi S)\left(\frac{\sin\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda} + \phi\right)}{p} + \frac{\cos\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda - \pi S)p^2} - \frac{\sin\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda}\right)\cos(\phi)}{p} - \frac{\cos\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda}\right)\sin(\phi)}{p}\right) / \lambda + \frac{1}{4} S (g-1)$$

$$(\pi S g - k\lambda - \pi S)\left(\frac{\sin\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda} - \phi\right)}{p} + \frac{\cos\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda} - \phi\right)\lambda}{(\pi S g - k\lambda - \pi S)p^2} - \frac{\sin\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\cos(\phi)}{p} + \frac{\cos\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\sin(\phi)}{p}\right) / \lambda + \frac{1}{4} S (1+g)$$

$$(\pi S g + k\lambda + \pi S)\left(\frac{\sin\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda} + \phi\right)}{p} + \frac{\cos\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda} + \phi\right)\lambda}{(\pi S g + k\lambda + \pi S)p^2} - \frac{\sin\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda}\right)\cos(\phi)}{p} - \frac{\cos\left(\frac{(\pi S g + k\lambda + \pi S)p}{\lambda}\right)\sin(\phi)}{p}\right) / \lambda + \frac{1}{4} S g$$

$$(2\pi S g - k\lambda)\left(\frac{\sin\left(\frac{(2\pi S g - k\lambda)p}{\lambda} - \phi\right)}{p} + \frac{\cos\left(\frac{(2\pi S g - k\lambda)p}{\lambda} - \phi\right)\lambda}{(2\pi S g - k\lambda)p^2} - \frac{\sin\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\cos(\phi)}{p} + \frac{\cos\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right) / \lambda - \frac{1}{4} S g$$

$$(2\pi S g + k\lambda)\left(\frac{\sin\left(\frac{(2\pi S g + k\lambda)p}{\lambda} + \phi\right)}{p} + \frac{\cos\left(\frac{(2\pi S g + k\lambda)p}{\lambda} + \phi\right)\lambda}{(2\pi S g + k\lambda)p^2} - \frac{\sin\left(\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\cos(\phi)}{p} - \frac{\cos\left(\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right) / \lambda$$

$$- \frac{\lambda k^2 \left(\frac{\sin(k p + \phi)}{k^2 p^3} + \frac{1}{2}\frac{\sin(k p + \phi)}{p} - \frac{1}{2}\frac{\sin(k p)\cos(\phi)}{p} - \frac{1}{2}\frac{\cos(k p)\sin(\phi)}{p}\right)}{\pi} + \frac{1}{4}$$

$$(2\pi S+k\lambda)^2\left(\frac{\sin\left(\frac{(2\pi S+k\lambda)p}{\lambda}+\phi\right)\lambda^2}{(2\pi S+k\lambda)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(2\pi S+k\lambda)p}{\lambda}+\phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S+k\lambda)p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{(2\pi S+k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)-\frac{1}{4}$$

$$(2\pi S-k\lambda)^2\left(\frac{\sin\left(\frac{(2\pi S-k\lambda)p}{\lambda}-\phi\right)\lambda^2}{(2\pi S-k\lambda)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(2\pi S-k\lambda)p}{\lambda}-\phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S-k\lambda)p}{\lambda}\right)\cos(\phi)}{p}+\frac{1}{2}\frac{\cos\left(\frac{(2\pi S-k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)+\frac{1}{2}$$

$$(\pi S g+k\lambda-\pi S)^2\left(\frac{\sin\left(\frac{(\pi S g+k\lambda-\pi S)p}{\lambda}+\phi\right)\lambda^2}{(\pi S g+k\lambda-\pi S)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(\pi S g+k\lambda-\pi S)p}{\lambda}+\phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(\pi S g+k\lambda-\pi S)p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{(\pi S g+k\lambda-\pi S)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)+\frac{1}{2}$$

$$(\pi S g-k\lambda+\pi S)^2\left(\frac{\sin\left(\frac{(\pi S g-k\lambda+\pi S)p}{\lambda}-\phi\right)\lambda^2}{(\pi S g-k\lambda+\pi S)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(\pi S g-k\lambda+\pi S)p}{\lambda}-\phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(\pi S g-k\lambda+\pi S)p}{\lambda}\right)\cos(\phi)}{p}+\frac{1}{2}\frac{\cos\left(\frac{(\pi S g-k\lambda+\pi S)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)-\frac{1}{2}$$

$$(\pi S g+k\lambda+\pi S)^2\left(\frac{\sin\left(\frac{(\pi S g+k\lambda+\pi S)p}{\lambda}+\phi\right)\lambda^2}{(\pi S g+k\lambda+\pi S)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(\pi S g+k\lambda+\pi S)p}{\lambda}+\phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(\pi S g+k\lambda+\pi S)p}{\lambda}\right)\cos(\phi)}{p}-\frac{1}{2}\frac{\cos\left(\frac{(\pi S g+k\lambda+\pi S)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)-\frac{1}{2}$$

$$(\pi S g-k\lambda-\pi S)^2\left(\frac{\sin\left(\frac{(\pi S g-k\lambda-\pi S)p}{\lambda}-\phi\right)\lambda^2}{(\pi S g-k\lambda-\pi S)^2 p^3}+\frac{1}{2}\frac{\sin\left(\frac{(\pi S g-k\lambda-\pi S)p}{\lambda}-\phi\right)}{p}\right.$$

$$-\frac{1}{2}\frac{\sin\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\cos(\phi)}{p} + \frac{1}{2}\frac{\cos\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right)\sin(\phi)}{p}\Bigg)/(\pi\lambda) + \frac{1}{4}$$

$$(2\pi S g + k\lambda)^2\left(\frac{\sin\left(\frac{(2\pi S g + k\lambda)p}{\lambda} + \phi\right)\lambda^2}{(2\pi S g + k\lambda)^2 p^3} + \frac{1}{2}\frac{\sin\left(\frac{(2\pi S g + k\lambda)p}{\lambda} + \phi\right)}{p}\right.$$

$$\left.-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\cos(\phi)}{p} - \frac{1}{2}\frac{\cos\left(\frac{(2\pi S g + k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda) - \frac{1}{4}$$

$$(2\pi S g - k\lambda)^2\left(\frac{\sin\left(\frac{(2\pi S g - k\lambda)p}{\lambda} - \phi\right)\lambda^2}{(2\pi S g - k\lambda)^2 p^3} + \frac{1}{2}\frac{\sin\left(\frac{(2\pi S g - k\lambda)p}{\lambda} - \phi\right)}{p}\right.$$

$$\left.\left.-\frac{1}{2}\frac{\sin\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\cos(\phi)}{p} + \frac{1}{2}\frac{\cos\left(\frac{(2\pi S g - k\lambda)p}{\lambda}\right)\sin(\phi)}{p}\right)/(\pi\lambda)\right)/\pi$$

cost(%)

606 *multiplications* + 175 *divisions* + 249 *additions* + 102 *functions*

This had better be equal to the original integrand.

expand(simplify(*%% – integrand*))

0

Whew. However, this means only that Maple is able to successfully differentiate the result of an integration. It does not necessarily mean the integration was correct, though it increases that probability.

3. Another Approach to Evaluating the Integral.

3.1. Simplification of the Indefinite Integral.

Start with the indefinite integration result and then insert the limits for p. First, let's simplify the form of the indefinite integral result. Recall that the indefinite integral from the last section is

Iv_indef := collect(*Iv_indef*, [sin(ϕ), cos(ϕ), *p*, λ, *g*], *factor*)

$$Iv_indef := \left(-\frac{1}{8}k^2\left(2\,\mathrm{Ci}\left(\frac{(\pi S g - k\lambda - \pi S)p}{\lambda}\right) + 2\,\mathrm{Ci}\left(\frac{(\pi S g + k\lambda - \pi S)p}{\lambda}\right)\right.\right.$$

$$-2\,\mathrm{Ci}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right) + \mathrm{Ci}\left(\frac{(2\pi S g + k\lambda)\,p}{\lambda}\right) + \mathrm{Ci}\left(\frac{(2\pi S - k\lambda)\,p}{\lambda}\right)$$

$$+\mathrm{Ci}\left(\frac{(2\pi S + k\lambda)\,p}{\lambda}\right) - 2\,\mathrm{Ci}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right) - 4\,\mathrm{Ci}(k\,p)$$

$$+\mathrm{Ci}\left(\frac{(2\pi S g - k\lambda)\,p}{\lambda}\right)\Bigg)\lambda^2 \Big/ \pi^2 + \Bigg(-\frac{1}{4} S k \Bigg(\mathrm{Ci}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right)$$

$$+\mathrm{Ci}\left(\frac{(\pi S g + k\lambda - \pi S)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(\pi S g - k\lambda - \pi S)\,p}{\lambda}\right)$$

$$+\mathrm{Ci}\left(\frac{(2\pi S g + k\lambda)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(2\pi S g - k\lambda)\,p}{\lambda}\right)\Bigg) g / \pi - \frac{1}{4} S k \Bigg($$

$$\mathrm{Ci}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(\pi S g + k\lambda - \pi S)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(2\pi S - k\lambda)\,p}{\lambda}\right)$$

$$+\mathrm{Ci}\left(\frac{(\pi S g - k\lambda - \pi S)\,p}{\lambda}\right) + \mathrm{Ci}\left(\frac{(2\pi S + k\lambda)\,p}{\lambda}\right) - \mathrm{Ci}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right)\Bigg) / \pi\Bigg)\lambda$$

$$\Bigg)\sin(\phi) + \Bigg(\frac{1}{8} k^2 \Bigg(-\mathrm{Si}\left(\frac{(2\pi S + k\lambda)\,p}{\lambda}\right) + 4\,\mathrm{Si}(k\,p) + 2\,\mathrm{Si}\left(\frac{(\pi S g - k\lambda - \pi S)\,p}{\lambda}\right)$$

$$-2\,\mathrm{Si}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right) + 2\,\mathrm{Si}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right)$$

$$-2\,\mathrm{Si}\left(\frac{(\pi S g + k\lambda - \pi S)\,p}{\lambda}\right) - \mathrm{Si}\left(\frac{(2\pi S g + k\lambda)\,p}{\lambda}\right) + \mathrm{Si}\left(\frac{(2\pi S - k\lambda)\,p}{\lambda}\right)$$

$$+\mathrm{Si}\left(\frac{(2\pi S g - k\lambda)\,p}{\lambda}\right)\Bigg)\lambda^2 \Big/ \pi^2 + \Bigg(\frac{1}{4} S k \Bigg(\mathrm{Si}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right)$$

$$-\mathrm{Si}\left(\frac{(\pi S g + k\lambda - \pi S)\,p}{\lambda}\right) - \mathrm{Si}\left(\frac{(\pi S g - k\lambda - \pi S)\,p}{\lambda}\right) + \mathrm{Si}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right)$$

$$-\mathrm{Si}\left(\frac{(2\pi S g + k\lambda)\,p}{\lambda}\right) - \mathrm{Si}\left(\frac{(2\pi S g - k\lambda)\,p}{\lambda}\right)\Bigg) g / \pi - \frac{1}{4} S k \Bigg($$

$$-\mathrm{Si}\left(\frac{(\pi S g - k\lambda + \pi S)\,p}{\lambda}\right) - \mathrm{Si}\left(\frac{(\pi S g + k\lambda - \pi S)\,p}{\lambda}\right) + \mathrm{Si}\left(\frac{(2\pi S - k\lambda)\,p}{\lambda}\right)$$

$$-\mathrm{Si}\left(\frac{(\pi S g - k\lambda - \pi S)\,p}{\lambda}\right) + \mathrm{Si}\left(\frac{(2\pi S + k\lambda)\,p}{\lambda}\right) - \mathrm{Si}\left(\frac{(\pi S g + k\lambda + \pi S)\,p}{\lambda}\right)\Bigg) / \pi\Bigg)\lambda\Bigg)$$

$$\cos(\phi) - \frac{1}{8} k \Bigg(-2\cos\left(\frac{p\pi S g - p k\lambda + p\pi S - \phi\lambda}{\lambda}\right)$$

$$-2\cos\left(\frac{p\pi S g + p k\lambda + p\pi S + \phi\lambda}{\lambda}\right) + 2\cos\left(\frac{p\pi S g - p k\lambda - p\pi S - \phi\lambda}{\lambda}\right)$$

$$+\cos\left(\frac{2p\pi S - p k\lambda - \phi\lambda}{\lambda}\right) + 2\cos\left(\frac{p\pi S g + p k\lambda - p\pi S + \phi\lambda}{\lambda}\right)$$

$$+\cos\left(\frac{2\,p\,\pi\,S\,g+p\,k\,\lambda+\phi\,\lambda}{\lambda}\right)+\cos\left(\frac{2\,p\,\pi\,S+p\,k\,\lambda+\phi\,\lambda}{\lambda}\right)-4\cos(k\,p+\phi)$$

$$+\cos\left(\frac{2\,p\,\pi\,S\,g-p\,k\,\lambda-\phi\,\lambda}{\lambda}\right)\Bigg)\lambda^2\Big/(\pi^2\,p)-\frac{1}{8}\Bigg(\Bigg(-4\sin(k\,p+\phi)$$

$$-\sin\left(\frac{2\,p\,\pi\,S-p\,k\,\lambda-\phi\,\lambda}{\lambda}\right)+2\sin\left(\frac{p\,\pi\,S\,g+p\,k\,\lambda-p\,\pi\,S+\phi\,\lambda}{\lambda}\right)$$

$$-2\sin\left(\frac{p\,\pi\,S\,g-p\,k\,\lambda-p\,\pi\,S-\phi\,\lambda}{\lambda}\right)+2\sin\left(\frac{p\,\pi\,S\,g-p\,k\,\lambda+p\,\pi\,S-\phi\,\lambda}{\lambda}\right)$$

$$-2\sin\left(\frac{p\,\pi\,S\,g+p\,k\,\lambda+p\,\pi\,S+\phi\,\lambda}{\lambda}\right)+\sin\left(\frac{2\,p\,\pi\,S+p\,k\,\lambda+\phi\,\lambda}{\lambda}\right)$$

$$-\sin\left(\frac{2\,p\,\pi\,S\,g-p\,k\,\lambda-\phi\,\lambda}{\lambda}\right)+\sin\left(\frac{2\,p\,\pi\,S\,g+p\,k\,\lambda+\phi\,\lambda}{\lambda}\right)\Bigg)\lambda^2\Bigg)\Big/(\pi^2\,p^2)$$

Once again, we make the substitution $f=\frac{k\,\lambda}{\pi\,S}$. Additionally, we make the substitution $Q=\frac{k\,p}{f}$. Q replaces p as our independent variable. Here is a procedure that does the substitutions in the function arguments (*sin*, *cos*, *Ci*, *Si*).

$$\mathrm{collect}\left(\mathrm{subs}\left(\lambda=\frac{f\,\pi\,S}{k},\ k=\frac{Q\,f}{p},\ Iv_indef\right),\ [S,\ Q,\ \sin,\ \cos,\ \mathrm{Si},\ \mathrm{Ci}],\ factor\right)$$

$$\Bigg(\Bigg(\frac{1}{4}f\,(g-f-1)\,\mathrm{Ci}\left(\frac{(\pi\,S\,g-f\,\pi\,S-\pi\,S)\,Q}{\pi\,S}\right)$$

$$-\frac{1}{4}f\,(g+f-1)\,\mathrm{Ci}\left(\frac{(\pi\,S\,g+f\,\pi\,S-\pi\,S)\,Q}{\pi\,S}\right)-\frac{1}{8}f\,(f+2\,g)\,\mathrm{Ci}\left(\frac{(2\,\pi\,S\,g+f\,\pi\,S)\,Q}{\pi\,S}\right)$$

$$-\frac{1}{8}f\,(-2+f)\,\mathrm{Ci}\left(\frac{(2\,\pi\,S-f\,\pi\,S)\,Q}{\pi\,S}\right)-\frac{1}{8}f\,(f+2)\,\mathrm{Ci}\left(\frac{(2\,\pi\,S+f\,\pi\,S)\,Q}{\pi\,S}\right)$$

$$+\frac{1}{4}f\,(f+1+g)\,\mathrm{Ci}\left(\frac{(\pi\,S\,g+f\,\pi\,S+\pi\,S)\,Q}{\pi\,S}\right)+\frac{1}{2}f^2\,\mathrm{Ci}(Q\,f)$$

$$+\frac{1}{8}f\,(2\,g-f)\,\mathrm{Ci}\left(\frac{(2\,\pi\,S\,g-f\,\pi\,S)\,Q}{\pi\,S}\right)-\frac{1}{4}f\,(g-f+1)\,\mathrm{Ci}\left(\frac{(\pi\,S\,g-f\,\pi\,S+\pi\,S)\,Q}{\pi\,S}\right)\Bigg)$$

$$\sin(\phi)+\Bigg(-\frac{1}{4}f\,(g+f-1)\,\mathrm{Si}\left(\frac{(\pi\,S\,g+f\,\pi\,S-\pi\,S)\,Q}{\pi\,S}\right)$$

$$-\frac{1}{8}f\,(f+2\,g)\,\mathrm{Si}\left(\frac{(2\,\pi\,S\,g+f\,\pi\,S)\,Q}{\pi\,S}\right)+\frac{1}{8}f\,(-2+f)\,\mathrm{Si}\left(\frac{(2\,\pi\,S-f\,\pi\,S)\,Q}{\pi\,S}\right)$$

$$-\frac{1}{8}f\,(2\,g-f)\,\mathrm{Si}\left(\frac{(2\,\pi\,S\,g-f\,\pi\,S)\,Q}{\pi\,S}\right)+\frac{1}{4}f\,(g-f+1)\,\mathrm{Si}\left(\frac{(\pi\,S\,g-f\,\pi\,S+\pi\,S)\,Q}{\pi\,S}\right)$$

$$+\frac{1}{4}f\,(f+1+g)\,\mathrm{Si}\left(\frac{(\pi\,S\,g+f\,\pi\,S+\pi\,S)\,Q}{\pi\,S}\right)-\frac{1}{8}f\,(f+2)\,\mathrm{Si}\left(\frac{(2\,\pi\,S+f\,\pi\,S)\,Q}{\pi\,S}\right)$$

$$+\frac{1}{2}f^2\,\mathrm{Si}(Q\,f)-\frac{1}{4}f\,(g-f-1)\,\mathrm{Si}\left(\frac{(\pi\,S\,g-f\,\pi\,S-\pi\,S)\,Q}{\pi\,S}\right)\Bigg)\cos(\phi)+\Bigg($$

$$-\frac{1}{4}f\cos\left(\frac{\left(p\,\pi\,S\,g+p\,f\,\pi\,S-p\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{8}f\cos\left(\frac{\left(2\,p\,\pi\,S\,g+p\,f\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{4}f\cos\left(\frac{\left(p\,\pi\,S\,g-p\,f\,\pi\,S-p\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{8}f\cos\left(\frac{\left(2\,p\,\pi\,S-p\,f\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)-\frac{1}{8}f\cos\left(\frac{\left(2\,p\,\pi\,S+p\,f\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$+\frac{1}{2}f\cos(Q\,f+\phi)-\frac{1}{8}f\cos\left(\frac{\left(2\,p\,\pi\,S\,g-p\,f\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$+\frac{1}{4}f\cos\left(\frac{\left(p\,\pi\,S\,g-p\,f\,\pi\,S+p\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$+\frac{1}{4}f\cos\left(\frac{\left(p\,\pi\,S\,g+p\,f\,\pi\,S+p\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)\Bigg)/\,Q+\Bigg(\frac{1}{2}\sin(Q\,f+\phi)$$

$$+\frac{1}{8}\sin\left(\frac{\left(2\,p\,\pi\,S-p\,f\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{4}\sin\left(\frac{\left(p\,\pi\,S\,g+p\,f\,\pi\,S-p\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$+\frac{1}{4}\sin\left(\frac{\left(p\,\pi\,S\,g-p\,f\,\pi\,S-p\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{4}\sin\left(\frac{\left(p\,\pi\,S\,g-p\,f\,\pi\,S+p\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$+\frac{1}{4}\sin\left(\frac{\left(p\,\pi\,S\,g+p\,f\,\pi\,S+p\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{8}\sin\left(\frac{\left(2\,p\,\pi\,S+p\,f\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)+\frac{1}{8}\sin\left(\frac{\left(2\,p\,\pi\,S\,g-p\,f\,\pi\,S-\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)$$

$$-\frac{1}{8}\sin\left(\frac{\left(2\,p\,\pi\,S\,g+p\,f\,\pi\,S+\frac{\phi\,\pi\,S\,p}{Q}\right)Q}{\pi\,S\,p}\right)\Bigg)\Big/\,Q^2\Bigg)S^2$$

collect(termfunc(*%*, *factor*), [*S*, *Q*, sin, cos, Si, Ci], *factor*)

$$\Bigg(\Bigg(-\frac{1}{4}f\,(g-f+1)\,\mathrm{Ci}((g-f+1)\,Q)+\frac{1}{8}f\,(2\,g-f)\,\mathrm{Ci}((2\,g-f)\,Q)$$

$$+\frac{1}{4}f\,(f+1+g)\,\mathrm{Ci}((f+1+g)\,Q)-\frac{1}{8}f\,(f+2)\,\mathrm{Ci}((f+2)\,Q)$$

$$-\frac{1}{8}f\,(-2+f)\,\mathrm{Ci}(-(-2+f)\,Q)-\frac{1}{8}f\,(f+2\,g)\,\mathrm{Ci}((f+2\,g)\,Q)$$

$$-\frac{1}{4}f\,(g+f-1)\,\mathrm{Ci}((g+f-1)\,Q)+\frac{1}{4}f\,(g-f-1)\,\mathrm{Ci}((g-f-1)\,Q)+\frac{1}{2}f^2\,\mathrm{Ci}(Q\,f)\Bigg)$$

$$\sin(\phi)+\Bigg(-\frac{1}{8}f\,(f+2\,g)\,\mathrm{Si}((f+2\,g)\,Q)-\frac{1}{4}f\,(g+f-1)\,\mathrm{Si}((g+f-1)\,Q)$$

$$-\frac{1}{4}f\,(g-f-1)\,\mathrm{Si}((g-f-1)\,Q)-\frac{1}{8}f\,(f+2)\,\mathrm{Si}((f+2)\,Q)$$

$$+\frac{1}{4}f\,(f+1+g)\,\mathrm{Si}((f+1+g)\,Q)+\frac{1}{4}f\,(g-f+1)\,\mathrm{Si}((g-f+1)\,Q)$$

$$-\frac{1}{8}f\,(2\,g-f)\,\mathrm{Si}((2\,g-f)\,Q)-\frac{1}{8}f\,(-2+f)\,\mathrm{Si}((-2+f)\,Q)+\frac{1}{2}f^2\,\mathrm{Si}(Q\,f)\Bigg)\cos(\phi)+\Bigg($$

$$-\frac{1}{8}f\cos(-2\,Q+Q\,f+\phi)-\frac{1}{8}f\cos(2\,g\,Q+Q\,f+\phi)-\frac{1}{8}f\cos(2\,g\,Q-Q\,f-\phi)$$

$$+\frac{1}{2}f\cos(Q\,f+\phi)-\frac{1}{4}f\cos(g\,Q-Q\,f-Q-\phi)+\frac{1}{4}f\cos(g\,Q+Q\,f+Q+\phi)$$

$$-\frac{1}{4}f\cos(g\,Q+Q\,f-Q+\phi)+\frac{1}{4}f\cos(g\,Q-Q\,f+Q-\phi)-\frac{1}{8}f\cos(2\,Q+Q\,f+\phi)\Bigg)/\,Q$$

$$+\Bigg(-\frac{1}{8}\sin(2\,g\,Q+Q\,f+\phi)-\frac{1}{4}\sin(g\,Q+Q\,f-Q+\phi)+\frac{1}{4}\sin(g\,Q-Q\,f-Q-\phi)$$

$$-\frac{1}{4}\sin(g\,Q-Q\,f+Q-\phi)-\frac{1}{8}\sin(-2\,Q+Q\,f+\phi)+\frac{1}{2}\sin(Q\,f+\phi)$$

$$-\frac{1}{8}\sin(2\,Q+Q\,f+\phi)+\frac{1}{8}\sin(2\,g\,Q-Q\,f-\phi)+\frac{1}{4}\sin(g\,Q+Q\,f+Q+\phi)\Big)\Big/\,Q^2\Big)S^2$$

Iv_indef := %

3.2. Substitution of the Integration Limits into the Indefinite Integral Expression.

Now do the integration limits $Q=-\infty$ to $Q=\infty$.

$I1 := \text{collect}(\lim_{Q\to(-\infty)} Iv_indef, [\sin, \cos, \pi, f])$

$$\begin{aligned}
I1 := \Bigg(\Bigg(&-\frac{1}{16}\,\mathrm{signum}(f+2)+\frac{1}{8}\,\mathrm{signum}(f+1+g)+\frac{1}{16}\,\mathrm{signum}(-2+f)\\
&+\frac{1}{8}\,\mathrm{signum}(g-f+1)+\frac{1}{4}\,\mathrm{signum}(f)-\frac{1}{8}\,\mathrm{signum}(g-f-1)-\frac{1}{16}\,\mathrm{signum}(2\,g-f)\\
&-\frac{1}{8}\,\mathrm{signum}(g+f-1)-\frac{1}{16}\,\mathrm{signum}(f+2\,g)\Bigg)S^2 f^2+\Bigg(-\frac{1}{8}\,\mathrm{signum}(-2+f)\\
&+\frac{1}{8}(1+\mathrm{signum}(2\,g-f))\,g-\frac{1}{8}(1+\mathrm{signum}(g-f+1))\,g+\frac{1}{8}(1+\mathrm{signum}(g-f-1))\,g\\
&-\frac{1}{8}(1+\mathrm{signum}(g+f-1))\,g-\frac{1}{8}(1+\mathrm{signum}(f+2\,g))\,g+\frac{1}{8}(1+\mathrm{signum}(f+1+g))\,g\\
&-\frac{1}{8}\,\mathrm{signum}(g-f+1)+\frac{1}{8}\,\mathrm{signum}(f+1+g)-\frac{1}{8}\,\mathrm{signum}(g-f-1)\\
&+\frac{1}{8}\,\mathrm{signum}(g+f-1)-\frac{1}{8}\,\mathrm{signum}(f+2)\Bigg)S^2 f\Bigg)\pi\sin(\phi)+\Bigg(\Bigg(-\frac{1}{8}\,\mathrm{signum}(g-f-1)\\
&-\frac{1}{4}\,\mathrm{signum}(f)+\frac{1}{16}\,\mathrm{signum}(f+2\,g)+\frac{1}{16}\,\mathrm{signum}(f+2)-\frac{1}{16}\,\mathrm{signum}(2\,g-f)\\
&+\frac{1}{8}\,\mathrm{signum}(g-f+1)+\frac{1}{8}\,\mathrm{signum}(g+f-1)-\frac{1}{8}\,\mathrm{signum}(f+1+g)+\frac{1}{16}\,\mathrm{signum}(-2+f)\\
&\Bigg)S^2 f^2+\Bigg(-\frac{1}{8}\,\mathrm{signum}(g+f-1)-\frac{1}{8}\,\mathrm{signum}(g-f+1)-\frac{1}{8}\,\mathrm{signum}(g-f-1)\\
&-\frac{1}{8}\,\mathrm{signum}(g-f+1)\,g+\frac{1}{8}\,\mathrm{signum}(g-f-1)\,g-\frac{1}{8}\,\mathrm{signum}(f+1+g)\,g\\
&+\frac{1}{8}\,\mathrm{signum}(f+2\,g)\,g+\frac{1}{8}\,\mathrm{signum}(f+2)+\frac{1}{8}\,\mathrm{signum}(g+f-1)\,g-\frac{1}{8}\,\mathrm{signum}(-2+f)\\
&-\frac{1}{8}\,\mathrm{signum}(f+1+g)+\frac{1}{8}\,\mathrm{signum}(2\,g-f)\,g\Bigg)S^2 f\Bigg)\pi\cos(\phi)
\end{aligned}$$

$I2 := \text{collect}(\lim_{Q\to\infty} Iv_indef, [\sin(\phi), \cos(\phi), \pi, f])$

$$I2 := \left(\left(\frac{1}{8}\,\mathrm{signum}(g-f-1) - \frac{1}{4}\,\mathrm{signum}(f) + \frac{1}{16}\,\mathrm{signum}(f+2) + \frac{1}{16}\,\mathrm{signum}(f+2\,g)\right.\right.$$
$$- \frac{1}{16}\,\mathrm{signum}(-2+f) - \frac{1}{8}\,\mathrm{signum}(f+1+g) - \frac{1}{8}\,\mathrm{signum}(g-f+1) + \frac{1}{16}\,\mathrm{signum}(2\,g-f)$$
$$\left.+ \frac{1}{8}\,\mathrm{signum}(g+f-1)\right) S^2 f^2 + \left(-\frac{1}{8}\,\mathrm{signum}(f+1+g) + \frac{1}{8}\,(-1+\mathrm{signum}(g+f-1))\,g\right.$$
$$+ \frac{1}{8}\,\mathrm{signum}(g-f+1) + \frac{1}{8}\,(-1+\mathrm{signum}(f+2\,g))\,g + \frac{1}{8}\,\mathrm{signum}(f+2)$$
$$- \frac{1}{8}\,(-1+\mathrm{signum}(f+1+g))\,g + \frac{1}{8}\,(-1+\mathrm{signum}(g-f+1))\,g$$
$$- \frac{1}{8}\,(-1+\mathrm{signum}(2\,g-f))\,g + \frac{1}{8}\,\mathrm{signum}(-2+f) + \frac{1}{8}\,\mathrm{signum}(g-f-1)$$
$$\left.\left.- \frac{1}{8}\,\mathrm{signum}(g+f-1) - \frac{1}{8}\,(-1+\mathrm{signum}(g-f-1))\,g\right) S^2 f\right)\pi \sin(\phi) + \left(\left(\right.\right.$$
$$- \frac{1}{8}\,\mathrm{signum}(g-f+1) + \frac{1}{8}\,\mathrm{signum}(g-f-1) - \frac{1}{16}\,\mathrm{signum}(-2+f) - \frac{1}{16}\,\mathrm{signum}(f+2)$$
$$+ \frac{1}{8}\,\mathrm{signum}(f+1+g) - \frac{1}{8}\,\mathrm{signum}(g+f-1) - \frac{1}{16}\,\mathrm{signum}(f+2\,g) + \frac{1}{4}\,\mathrm{signum}(f)$$
$$\left.+ \frac{1}{16}\,\mathrm{signum}(2\,g-f)\right) S^2 f^2 + \left(\frac{1}{8}\,\mathrm{signum}(g-f+1)\,g + \frac{1}{8}\,\mathrm{signum}(f+1+g)\,g\right.$$
$$- \frac{1}{8}\,\mathrm{signum}(g-f-1)\,g + \frac{1}{8}\,\mathrm{signum}(f+1+g) + \frac{1}{8}\,\mathrm{signum}(-2+f) - \frac{1}{8}\,\mathrm{signum}(2\,g-f)\,g$$
$$+ \frac{1}{8}\,\mathrm{signum}(g-f+1) + \frac{1}{8}\,\mathrm{signum}(g-f-1) + \frac{1}{8}\,\mathrm{signum}(g+f-1)$$
$$\left.\left.- \frac{1}{8}\,\mathrm{signum}(f+2\,g)\,g - \frac{1}{8}\,\mathrm{signum}(f+2) - \frac{1}{8}\,\mathrm{signum}(g+f-1)\,g\right) S^2 f\right)\pi \cos(\phi)$$

We obtain the result

Iv2 := collect(algsubs($k\,\lambda = \pi\,f\,S$, simplify(*I2* − *I1*)), [S, π, sin, cos, signum], *factor*)

$$Iv2 := \left(\left(\frac{1}{8}\,f\,(f+2\,g)\,\mathrm{signum}(f+2\,g) + \frac{1}{8}\,f\,(f+2)\,\mathrm{signum}(f+2)\right.\right.$$
$$+ \frac{1}{4}\,f\,(g+f-1)\,\mathrm{signum}(g+f-1) + \frac{1}{4}\,f\,(g-f+1)\,\mathrm{signum}(g-f+1)$$
$$- \frac{1}{8}\,f\,(-2+f)\,\mathrm{signum}(-2+f) - \frac{1}{8}\,f\,(2\,g-f)\,\mathrm{signum}(2\,g-f)$$
$$\left.- \frac{1}{4}\,f\,(g-f-1)\,\mathrm{signum}(g-f-1) - \frac{1}{4}\,f\,(f+1+g)\,\mathrm{signum}(f+1+g) - \frac{1}{2}\,f\,|f|\right)\sin(\phi) +$$

$$\left(-\frac{1}{8} f(f+2g)\,\mathrm{signum}(f+2g)-\frac{1}{8} f(f+2)\,\mathrm{signum}(f+2)\right.$$
$$-\frac{1}{4} f(g+f-1)\,\mathrm{signum}(g+f-1)+\frac{1}{4} f(g-f+1)\,\mathrm{signum}(g-f+1)$$
$$-\frac{1}{8} f(-2+f)\,\mathrm{signum}(-2+f)-\frac{1}{8} f(2g-f)\,\mathrm{signum}(2g-f)$$
$$\left.-\frac{1}{4} f(g-f-1)\,\mathrm{signum}(g-f-1)+\frac{1}{4} f(f+1+g)\,\mathrm{signum}(f+1+g)+\frac{1}{2} f|f|\right)\cos(\phi)\Bigg)$$
$$\pi S^2$$

$$\mathrm{convert}\left(\frac{\%}{\pi S^2}, \mathrm{abs}\right)$$

$$\left(\frac{1}{8}\frac{f(f+2g)^2}{|f+2g|}+\frac{1}{8}\frac{f(f+2)^2}{|f+2|}+\frac{1}{4}\frac{f(g+f-1)^2}{|g+f-1|}+\frac{1}{4}\frac{f(g-f+1)^2}{|g-f+1|}-\frac{1}{8}\frac{f(-2+f)^2}{|-2+f|}\right.$$
$$\left.-\frac{1}{8}\frac{f(2g-f)^2}{|2g-f|}-\frac{1}{4}\frac{f(g-f-1)^2}{|g-f-1|}-\frac{1}{4}\frac{f(f+1+g)^2}{|f+1+g|}-\frac{1}{2} f|f|\right)\sin(\phi)+\left(-\frac{1}{8}\frac{f(f+2g)^2}{|f+2g|}\right.$$
$$-\frac{1}{8}\frac{f(f+2)^2}{|f+2|}-\frac{1}{4}\frac{f(g+f-1)^2}{|g+f-1|}+\frac{1}{4}\frac{f(g-f+1)^2}{|g-f+1|}-\frac{1}{8}\frac{f(-2+f)^2}{|-2+f|}-\frac{1}{8}\frac{f(2g-f)^2}{|2g-f|}$$
$$\left.-\frac{1}{4}\frac{f(g-f-1)^2}{|g-f-1|}+\frac{1}{4}\frac{f(f+1+g)^2}{|f+1+g|}+\frac{1}{2} f|f|\right)\cos(\phi)$$

$L_1 := \mathrm{location}(\%, \mathrm{coeff}(\%, \sin(\phi), 1))$

$$L_1 := [1, 1]$$

$L_2 := \mathrm{location}(\%\%, \mathrm{coeff}(\%\%, \cos(\phi), 1))$

$$L_2 := [2, 1]$$

$\mathrm{subsop}(L_1 = \mathrm{map}(x \to \mathrm{simplify}(\mathrm{factor}(x)), \mathrm{coeff}(\%\%\%, \sin(\phi), 1)),$
$L_2 = \mathrm{map}(x \to \mathrm{simplify}(\mathrm{factor}(x)), \mathrm{coeff}(\%\%\%, \cos(\phi), 1)), \%\%\%)$

$$\left(\frac{1}{8} f|f+2g|+\frac{1}{8} f|f+2|+\frac{1}{4} f|g+f-1|+\frac{1}{4} f|g-f+1|-\frac{1}{8} f|-2+f|-\frac{1}{8}|2g-f|f\right.$$
$$\left.-\frac{1}{4}|g-f-1|f-\frac{1}{4} f|f+1+g|-\frac{1}{2} f|f|\right)\sin(\phi)+\left(-\frac{1}{8} f|f+2g|-\frac{1}{8} f|f+2|\right.$$
$$-\frac{1}{4} f|g+f-1|+\frac{1}{4} f|g-f+1|-\frac{1}{8} f|-2+f|-\frac{1}{8}|2g-f|f-\frac{1}{4}|g-f-1|f$$

$$+\frac{1}{4}f\left|f+1+g\right|+\frac{1}{2}f\left|f\right|\Bigg)\cos(\phi)$$

$$'Iv' = \frac{\pi f S^2 \operatorname{collect}\left(\frac{8\,\%}{f}, [\sin, \cos, f, g]\right)}{8}$$

$$Iv = \frac{1}{8}\pi f S^2 \left(\left(\left|f+2g\right|+\left|f+2\right|+2\left|g+f-1\right|+2\left|g-f+1\right|-\left|-2+f\right|-\left|2g-f\right| - 2\left|g-f-1\right|-2\left|f+1+g\right|-4\left|f\right|\right)\sin(\phi) + \left(-\left|f+2g\right|-\left|f+2\right|-2\left|g+f-1\right| + 2\left|g-f+1\right|-\left|-2+f\right|-\left|2g-f\right|-2\left|g-f-1\right|+2\left|f+1+g\right|+4\left|f\right|\right)\cos(\phi)\right)$$

$\operatorname{simplify}(\operatorname{rhs}(\%) - \operatorname{convert}(Iv2, \operatorname{abs}))$

$$0$$

$Iv2 := \operatorname{rhs}(\%\%)$

This is not the same as in section 1 — how dissapointing. At least it does not contain those suspicious *ln* terms.

$\Delta = \operatorname{collect}(\operatorname{simplify}(Iv2 - Iv), [S, \pi, \sin, \cos, \ln], factor)$

$$\Delta = \Bigg(\Bigg(-\frac{1}{8}f\left(2\left|f+1+g\right|-\left|f+2\right|-\left|f+2g\right|+4\left|f\right|\right)\sin(\phi) + \frac{1}{8}f\left(-4f+2\left|f+1+g\right|-\left|f+2\right|-\left|f+2g\right|+4\left|f\right|\right)\cos(\phi)\Bigg)\pi + \Bigg(\frac{1}{4}f(g-f+1)\ln(-g+f-1) + \frac{1}{4}f(g+f-1)\ln(-g-f+1) - \frac{1}{8}f(2g-f)\ln(-2g+f) - \frac{1}{4}f(g-f-1)\ln(-g+f+1) + \frac{1}{8}f(-2+f)\ln(-2+f) - \frac{1}{8}f(-2+f)\ln(2-f) - \frac{1}{4}f(g+f-1)\ln(g+f-1) + \frac{1}{4}f(g-f-1)\ln(g-f-1) + \frac{1}{8}f(2g-f)\ln(2g-f) - \frac{1}{4}f(g-f+1)\ln(g-f+1)\Bigg)\sin(\phi)\Bigg)S^2$$

The *sin* and *cos* parts are

$Iv2_sin := \operatorname{coeff}(Iv2, \sin(\phi))$

$$Iv2_sin := \frac{1}{8}\pi f S^2 \left(\left|f+2g\right|+\left|f+2\right|+2\left|g+f-1\right|+2\left|g-f+1\right|-\left|-2+f\right|-\left|2g-f\right| - 2\left|g-f-1\right|-2\left|f+1+g\right|-4\left|f\right|\right)$$

$Iv2_cos := \operatorname{coeff}(Iv2, \cos(\phi))$

$$Iv2_cos := \frac{1}{8}\pi f S^2 \left(-|f+2g| - |f+2| - 2|g+f-1| + 2|g-f+1| - |-2+f| - |2g-f| - 2|g-f-1| + 2|f+1+g| + 4|f|\right)$$

3.3. Characterization of the *sin* and *cos* Terms.

As before, define the coefficients of $\frac{\pi S^2 \sin(\phi)}{8}$ and $\frac{\pi S^2 \cos(\phi)}{8}$ as functions of g and f.

$$G2_s := \mathrm{fn}\left(\mathrm{collect}\left(\frac{8\, Iv2_sin}{\pi S^2}, [f, g]\right), g, f\right)$$

$$G2_c := \mathrm{fn}\left(\mathrm{collect}\left(\frac{8\, Iv2_cos}{\pi S^2}, [f, g]\right), g, f\right)$$

$$G2_c(g, f)$$

$$f\left(-|f+2g| - |f+2| - 2|g+f-1| + 2|g-f+1| - |-2+f| - |2g-f| - 2|g-f-1| + 2|f+1+g| + 4|f|\right)$$

$$G2_s(g, f)$$

$$f\left(|f+2g| + |f+2| + 2|g+f-1| + 2|g-f+1| - |-2+f| - |2g-f| - 2|g-f-1| - 2|f+1+g| - 4|f|\right)$$

For $g = 0$ these of course reduce to much simpler expressions. The *cos* term becomes

$$G2_c(0, f)$$

$$f\left(2|f| - |f+2| - |-2+f|\right)$$

This is identical to eq. (11) in TM97-01. For $2 < |f|$ this expression is identically zero. The *sin* term becomes

$$G2_s(0, f)$$

$$f\left(-4|f| + |f+2| + 4|f-1| - |-2+f| - 4|f+1|\right)$$

Let's look at some plots.

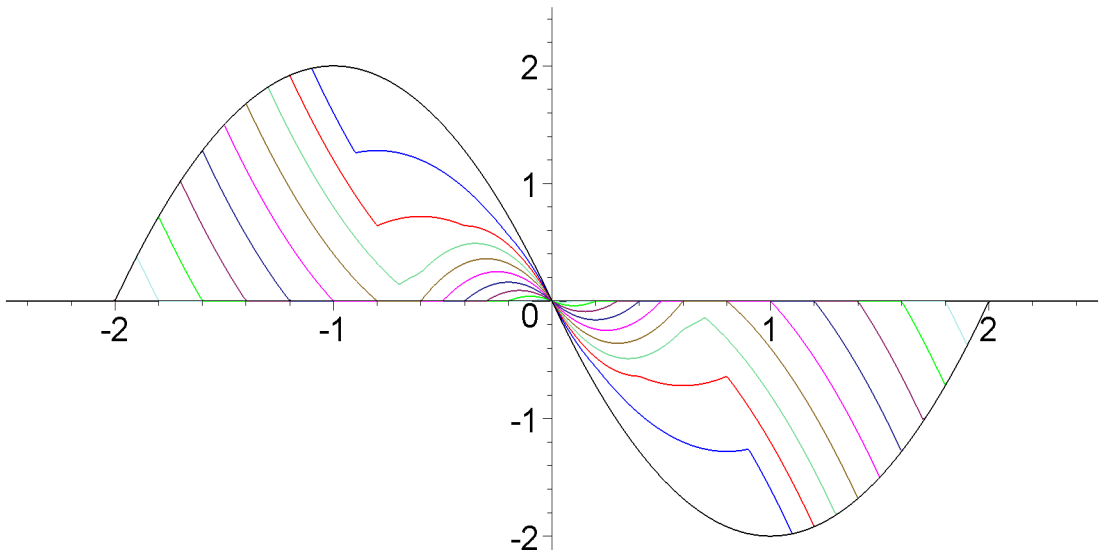
```
Gplot := proc( F, vals::list, frange::range )
  local p, k, Grange;
  if nargs=3 then
    Grange := args[3];
  else
    Grange := -2.5..2.5;
  fi;
  p := [];
  for k from 1 to nops(gvals) do
    p := [ op(p), plot( F(vals[k],f), f=frange,
                        color=mycolors[(k-1 mod 10)+1],
```

```
                                view=[frange,Grange], numpoints=200,
                                thickness=2 ) ];
  od;
  plots[display](p,axes=normal);
end:
```

$gvals := [\mathrm{seq}(.1\, i, i = 0 .. 10)]$

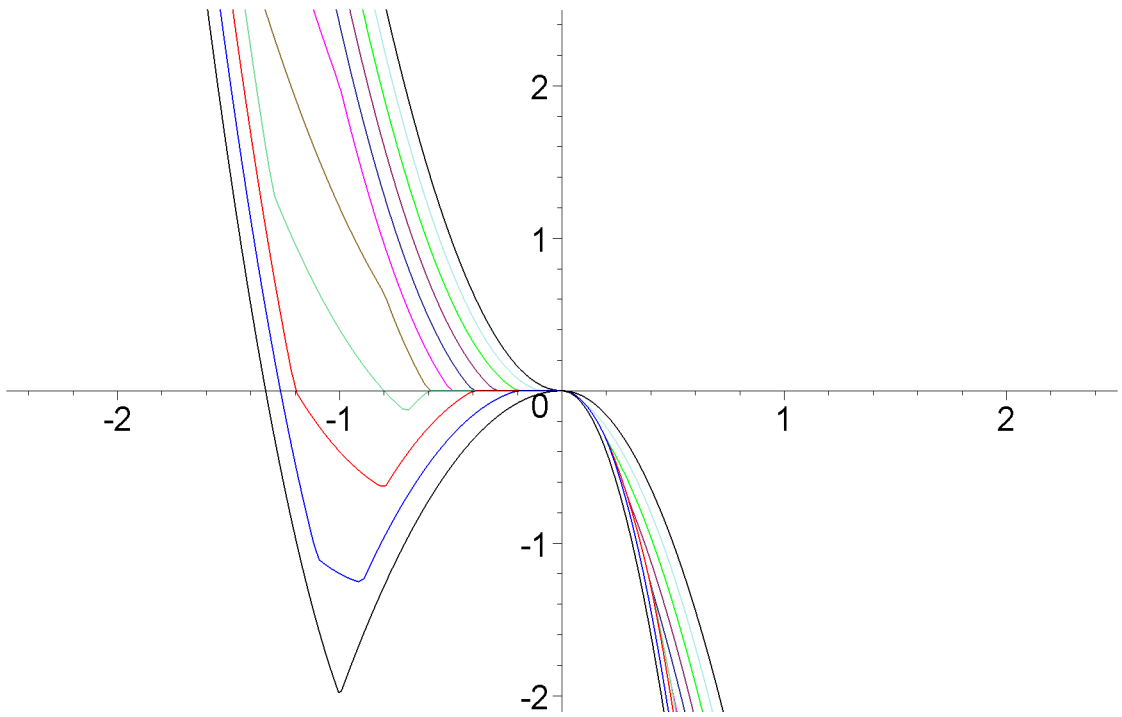
Plot the *cos* term only:

$\mathrm{Gplot}(G2_c, gvals, -2.5 .. 2.5)$



These curves appear the same as those plotted in TM97-01. Hence, the function G_c is probably identical to the *G* function in Appendix A of TM97-01. Now plot the *sin* term only:

$\mathrm{Gplot}(G2_s, gvals, -2.5 .. 2.5)$



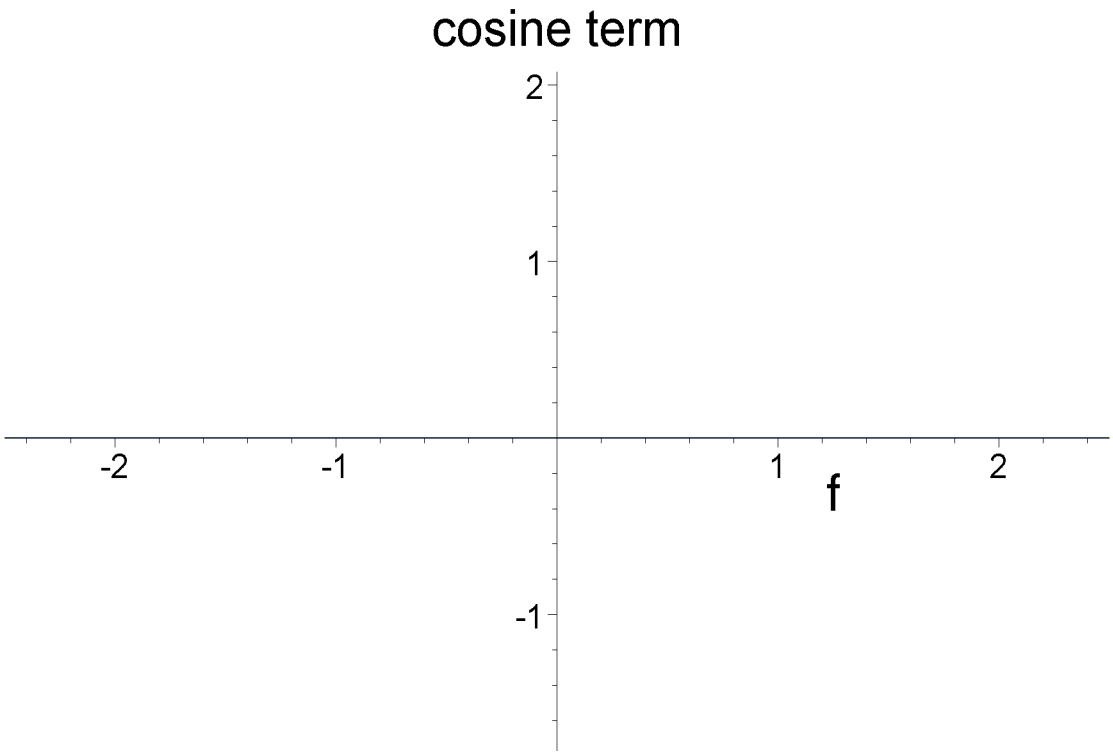
Oh my. The sin term appears to have problems.

Animations for fun:

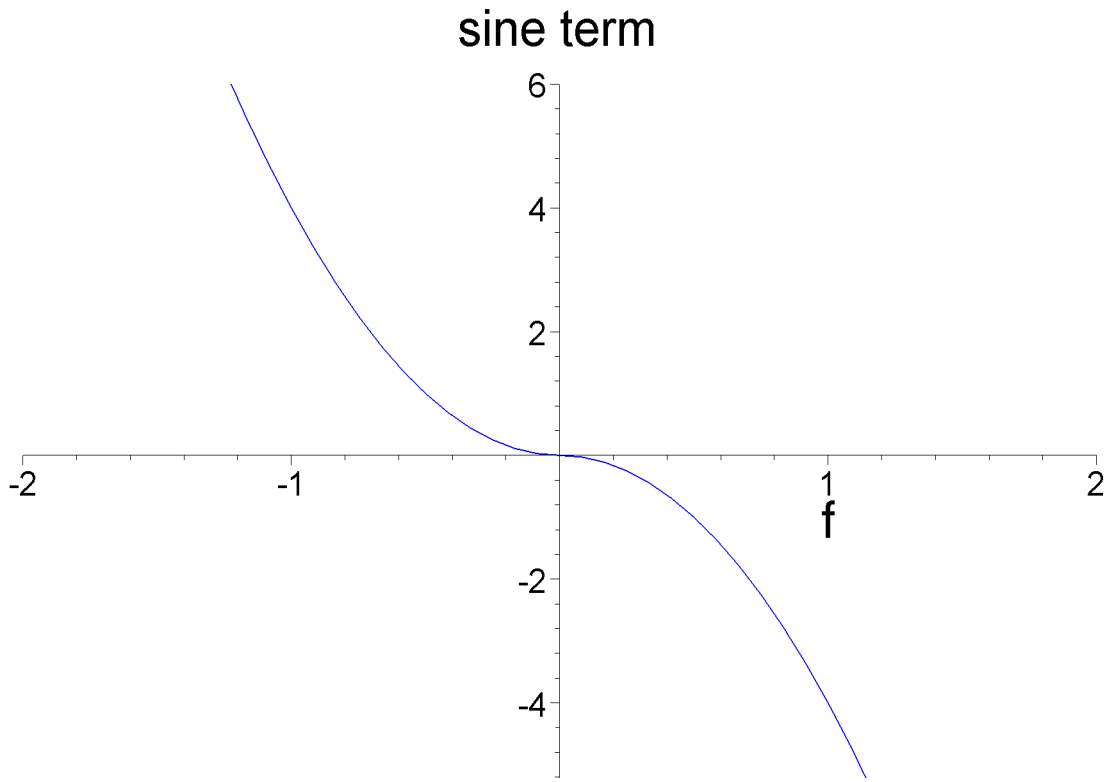
$G2_c(0, 1)$

-2

```
plots[animate]( G2[c](abs(t-1),f), f=-2.5..2.5, t=0..2, frames=25,
                color=blue, axes=normal, thickness=2,
                labels=["f",""], title="cosine term" );
```




```
plots[animate]( G2[s](abs(t-1),f), f=-2..2, t=0..2, frames=25,
                color=blue, axes=normal, thickness=2,
                view=[-2..2,-6..6],
                labels=["f",""], title="sine term" );
```



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