Rigid Body Dynamics of a Truncated Cone

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Euler Equations of Motion for a Rigid Body

```
restart;
alias( I=I, psi=psi(t), theta=theta(t), phi=phi(t),
       Omega[x]=Omega[x](t), Omega[y]=Omega[y](t), Omega[z]=Omega[z](t),
       Omega[phi]=Omega[phi](t), Omega[psi]=Omega[psi](t),
       Omega[theta]=Omega[theta](t) ):
```

In the frame of reference attached to the rotating body (the body frame), the equations of motion are

```
EulerEqs := mat(
   I[x]*diff(Omega[x],t) + (I[z]-I[y])*Omega[y]*Omega[z] - K[x],
   I[y]*diff(Omega[y],t) + (I[x]-I[z])*Omega[x]*Omega[z] - K[y],
   I[z]*diff(Omega[z],t) + (I[y]-I[x])*Omega[x]*Omega[y] - K[z] );
```

$$EulerEqs := \begin{bmatrix} I_x \left(\frac{\partial}{\partial t}\Omega_x\right) + (I_z - I_y)\,\Omega_y\,\Omega_z - K_x \\ I_y \left(\frac{\partial}{\partial t}\Omega_y\right) + (I_x - I_z)\,\Omega_x\,\Omega_z - K_y \\ I_z \left(\frac{\partial}{\partial t}\Omega_z\right) + (I_y - I_x)\,\Omega_x\,\Omega_y - K_z \end{bmatrix}$$

```
latex(EulerEqs,`d:/dynamics/precession/EulerEqs.tex`);
```

where I_x, I_y, I_z are the principal moments of inertia of the body, $\Omega_x, \Omega_y, \Omega_z$ are the angular velocities of the body about the principal axes, and K_x, K_y, K_z are the components of the torque acting on the body.

Eulerian Angle Transformation

Rotation Matrix

Construct a rotation matrix that transforms coordinates from the fixed frame (X,Y,Z) to the body frame (x,y,z). First, rotate the coordinates ccw around the Z axis.

```
r1 := matrix([[cos(phi),sin(phi),0],[-sin(phi),cos(phi),0],[0,0,1]]);
```

$$r1 := \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, rotate ccw around the X' axis.

```
r2 := matrix( [ [1,0,0], [0,cos(psi),sin(psi)], [0,-sin(psi),cos(psi)] ] );
```

$$r2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & \sin(\psi) \\ 0 & -\sin(\psi) & \cos(\psi) \end{bmatrix}$$

Next, rotate ccw around the Z" axis.

```
r3 :=
matrix([[cos(theta),sin(theta),0],[-sin(theta),cos(theta),0],[0,0,1]]);
```

$$r3 := \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now combine the rotations into a single rotation matrix.

```
R := (p,q,r)->evalm( subs( theta=r, eval(r3) ) &*
                     subs( psi=q,   eval(r2) ) &*
                     subs( phi=p,   eval(r1) ) ):
```

Hence, we have the coordinate transformation

```
mat(x,y,z) = R(phi,psi,theta) &* mat(X,Y,Z);
```

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$[\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi), \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi), \sin(\theta)\sin(\psi)]$$
$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi), -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$
$$\cos(\theta)\sin(\psi)]$$
$$[\sin(\psi)\sin(\phi), -\sin(\psi)\cos(\phi), \cos(\psi)] \,\&*\, \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

```
latex(",`d:/dynamics/precession/FixedToBody.tex`);
```

which, written out, is

```
evalm( " );
```

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$[(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\, X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\, Y$$

$$+\sin(\theta)\sin(\psi)\,Z]$$

$$[(-\sin(\theta)\cos(\phi)-\cos(\theta)\cos(\psi)\sin(\phi))\,X+(-\sin(\theta)\sin(\phi)+\cos(\theta)\cos(\psi)\cos(\phi))\,Y$$

$$+\cos(\theta)\sin(\psi)\,Z]$$

$$[\sin(\psi)\sin(\phi)\,X-\sin(\psi)\cos(\phi)\,Y+\cos(\psi)\,Z]$$

The diagram below illustrates the three rotations.

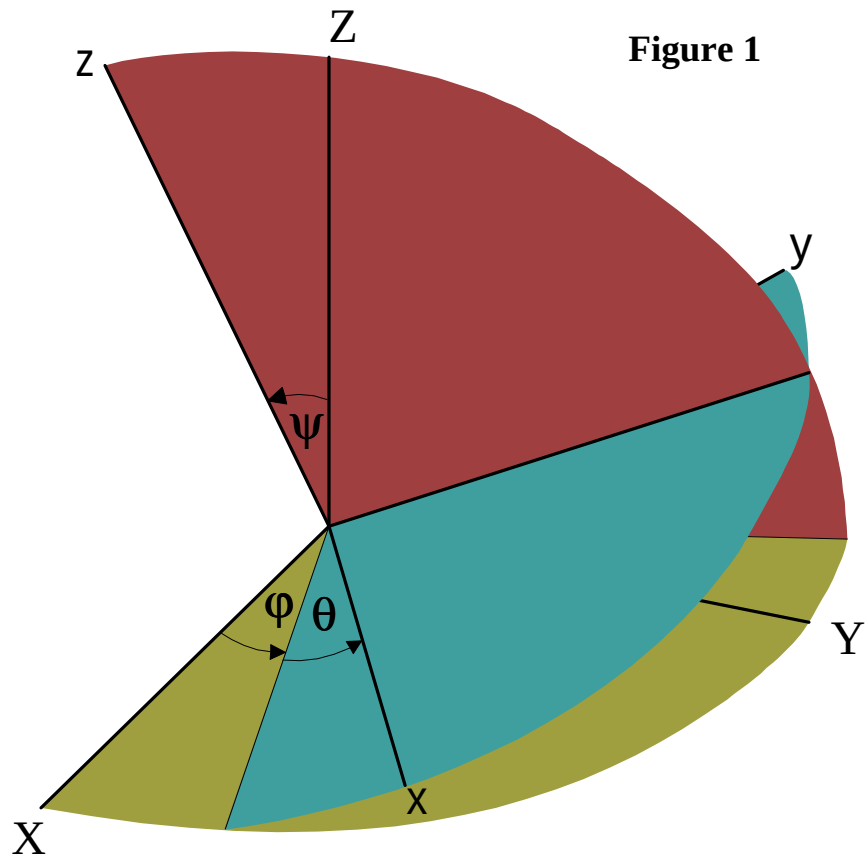


Figure 1

$$R(j\ ,y\ ,q) = R_Z(q)\,R_X(y\ )\,R_Z(j\ )$$

Angular Velocity Vector in the Body Frame

The angular velocity vector may be decomposed into components along each of the rotation axes used to construct the transformation matrix. If we transform those components to the body frame, then we can express the angular velocity vector in the body frame in terms of the Euler angles (ϕ, ψ, θ). The component of Ω along the first rotation axis, as viewed in the body frame, is

```
omega[phi] := evalm(R(0,psi,theta) &* mat( 0, 0, 1 )) * diff(phi,t);
```

$$\omega_\phi := \begin{bmatrix} \sin(\theta)\sin(\psi) \\ \cos(\theta)\sin(\psi) \\ \cos(\psi) \end{bmatrix} \left(\frac{\partial}{\partial t}\phi\right)$$

The component along the Y' axis is, in the body frame,

```
omega[psi] := evalm(R(0,0,theta) &* mat( 1, 0, 0 )) * diff(psi,t);
```

$$\omega_{\psi} := \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{bmatrix} \left(\frac{\partial}{\partial t} \psi \right)$$

Finally, the component along the Z" axis is simply

```
omega[theta] := mat( 0, 0, 1 ) * diff(theta,t);
```

$$\omega_{\theta} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\frac{\partial}{\partial t} \theta \right)$$

Hence, we have the angular velocity vector in the body frame,

```
Omega = mat(
evalm(omega[phi])[1,1]+evalm(omega[psi])[1,1]+evalm(omega[theta])[1,1],
evalm(omega[phi])[2,1]+evalm(omega[psi])[2,1]+evalm(omega[theta])[2,1],
evalm(omega[phi])[3,1]+evalm(omega[psi])[3,1]+evalm(omega[theta])[3,1] );
```

$$\Omega = \begin{bmatrix} \left(\frac{\partial}{\partial t} \phi \right) \sin(\theta) \sin(\psi) + \left(\frac{\partial}{\partial t} \psi \right) \cos(\theta) \\ \left(\frac{\partial}{\partial t} \phi \right) \cos(\theta) \sin(\psi) - \left(\frac{\partial}{\partial t} \psi \right) \sin(\theta) \\ \left(\frac{\partial}{\partial t} \phi \right) \cos(\psi) + \left(\frac{\partial}{\partial t} \theta \right) \end{bmatrix}$$

```
latex(",`d:/dynamics/precession/OmegaBodyFrame.tex`);
```

```
Obody := rhs("):
```

Rigid Body Equations — General Case

Equations of Motion

Substituting back into the Euler equations, we find

```
subs( Omega[x]=Obody[1,1], Omega[y]=Obody[2,1],
      Omega[z]=Obody[3,1], eval(EulerEqs) );
```

$$\left[I_x \left(\frac{\partial}{\partial t} \left(\left(\frac{\partial}{\partial t} \phi \right) \sin(\theta) \sin(\psi) + \left(\frac{\partial}{\partial t} \psi \right) \cos(\theta) \right) \right) + (I_z - I_y) \left(\left(\frac{\partial}{\partial t} \phi \right) \cos(\theta) \sin(\psi) - \left(\frac{\partial}{\partial t} \psi \right) \sin(\theta) \right) \left(\left(\frac{\partial}{\partial t} \phi \right) \cos(\psi) + \left(\frac{\partial}{\partial t} \theta \right) \right) - K_x \right]$$

$$\left[I_y \left(\frac{\partial}{\partial t} \left(\left(\frac{\partial}{\partial t} \phi \right) \cos(\theta) \sin(\psi) - \left(\frac{\partial}{\partial t} \psi \right) \sin(\theta) \right) \right) + (I_x - I_z) \left(\left(\frac{\partial}{\partial t} \phi \right) \sin(\theta) \sin(\psi) + \left(\frac{\partial}{\partial t} \psi \right) \cos(\theta) \right) \left(\left(\frac{\partial}{\partial t} \phi \right) \cos(\psi) + \left(\frac{\partial}{\partial t} \theta \right) \right) - K_y \right]$$

$$\left[I_z \left(\frac{\partial}{\partial t} \left(\left(\frac{\partial}{\partial t} \phi \right) \cos(\psi) + \left(\frac{\partial}{\partial t} \theta \right) \right) \right) \right.$$

$$+(I_y - I_x)\left(\left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta)\right)\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta)\right)$$
$$-K_z\Bigg]$$

```
foo := ":
```

```
eq1 := collect( foo[1,1]/I[x],
        [diff(psi,t,t),diff(phi,t,t),sin(theta),cos(theta),sin(psi),diff],
        factor );
```

$$eq1 := \left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)$$
$$+\left(\frac{\cos(\psi)(I_x - I_z + I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x} - \frac{(I_x + I_z - I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}\right)\sin(\theta)$$
$$+\left(\frac{(I_z - I_y)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_x} + \frac{(I_x + I_z - I_y)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\sin(\psi)\cos(\theta) - \frac{K_x}{I_x}$$

```
eq2 := collect( foo[2,1]/I[y],
        [diff(psi,t,t),diff(phi,t,t),sin(theta),cos(theta),sin(psi),diff],
        factor );
```

$$eq2 := -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)$$
$$+\left(\frac{(I_x - I_z)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_y} + \frac{(-I_y + I_x - I_z)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\sin(\psi)\sin(\theta)$$
$$+\left(\frac{\cos(\psi)(I_x - I_z + I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y} + \frac{(-I_y + I_x - I_z)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}\right)\cos(\theta) - \frac{K_y}{I_y}$$

```
eq3 := collect( cosfix(foo[3,1]/I[z],theta),
                [diff(theta,t,t),diff(phi,t,t),sin(psi),diff,sin],
                factor );
```

$$eq3 := \left(\frac{\partial^2}{\partial t^2}\theta\right) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi) - \frac{(-I_y + I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2}{I_z}$$
$$+\left(2\frac{(-I_y + I_x)\sin(\theta)^2}{I_z} - \frac{I_x + I_z - I_y}{I_z}\right)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)$$

$$+\frac{(-I_y+I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)^2}{I_z}-\frac{K_z}{I_z}$$

```
BodyEqs := [eq1,eq2,eq3]:
latex(BodyEqs,`d:/dynamics/precession/RigidBodyEqs.tex`);
mat(BodyEqs);
```

$$\left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)+\left(\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}-\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_x}\right)\sin(\theta)+\left(\frac{(I_z-I_y)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_x}+\frac{(I_x+I_z-I_y)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_x}\right)\sin(\psi)\cos(\theta)-\frac{K_x}{I_x}\right]$$

$$\left[-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\frac{(I_x-I_z)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_y}+\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}\right)\sin(\psi)\sin(\theta)+\left(\frac{\cos(\psi)(I_x-I_z+I_y)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_y}+\frac{(-I_y+I_x-I_z)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_y}\right)\cos(\theta)-\frac{K_y}{I_y}\right]$$

$$\left[\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\frac{(-I_y+I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\phi\right)^2\sin(\psi)^2}{I_z}+\left(2\frac{(-I_y+I_x)\sin(\theta)^2}{I_z}-\frac{I_x+I_z-I_y}{I_z}\right)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)+\frac{(-I_y+I_x)\cos(\theta)\sin(\theta)\left(\frac{\partial}{\partial t}\psi\right)^2}{I_z}-\frac{K_z}{I_z}\right]$$

Torque in the Body Frame

Conversion to a System of First-Order ODEs

Rigid Body Equations — Symmetric Top

General Case

Equations of Motion

```
SymTop := subs( I[x]=I[y], I[y]=I[xy], [eq1,eq2,eq3] );
```

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\cos(\psi)\,(2\,I_{xy}-I_z)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}-\frac{I_z\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}}\right)\sin(\theta) + \left(\frac{(I_z-I_{xy})\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_{xy}}+\frac{I_z\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\sin(\psi)\cos(\theta)-\frac{K_x}{I_{xy}}, -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\frac{(I_{xy}-I_z)\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2}{I_{xy}}-\frac{I_z\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}\right)\sin(\psi)\sin(\theta) + \left(\frac{\cos(\psi)\,(2\,I_{xy}-I_z)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)}{I_{xy}}-\frac{I_z\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)}{I_{xy}}\right)\cos(\theta)-\frac{K_y}{I_{xy}}, \left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)-\frac{K_z}{I_z}\right]$$

```
collect( simplify( SymTop[1], {1-I[z]/I[xy]=beta}, [I[z]] ),
[diff(psi,t,t),diff(phi,t,t),sin(theta),cos(theta),sin(psi),diff],
         factor );
```

$$\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\cos(\psi)\,(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\sin(\theta)$$

$$+\left(-\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta)-\frac{K_x}{I_{xy}}$$

```
collect( simplify( SymTop[2], {1-I[z]/I[xy]=beta}, [I[z]] ),
[diff(psi,t,t),diff(phi,t,t),sin(theta),cos(theta),sin(psi),diff],
        factor );
```

$$-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)$$
$$+\left(\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta)$$
$$+\left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\cos(\theta)-\frac{K_y}{I_{xy}}$$

```
CollectDiffs(simplify( SymTop[3], {1-I[z]/I[xy]=beta}, [I[z]] ));
```

$$\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\frac{K_z}{I_{xy}(-1+\beta)}$$

```
SymTop := ["""","","];
```

$$SymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)\right.$$
$$+\left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\sin(\theta)$$
$$+\left(-\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta)-\frac{K_x}{I_{xy}},-\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta)$$
$$+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta)$$
$$+\left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\cos(\theta)-\frac{K_y}{I_{xy}},$$
$$\left.\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)+\frac{K_z}{I_{xy}(-1+\beta)}\right]$$

```
latex( SymTop[1], `d:/dynamics/precession/SymTop1.tex` );
latex( SymTop[2], `d:/dynamics/precession/SymTop2.tex` );
latex( SymTop[3], `d:/dynamics/precession/SymTop3.tex` );
```

Conversion to a System of First-Order ODEs

We may write the equations of motion as a system of first-order differential equations.

```
subslist := [ diff(phi,t)=Omega[phi], diff(psi,t)=Omega[psi],
              diff(theta,t)=Omega[theta] ];
```

$$subslist := \left[\frac{\partial}{\partial t}\phi = \Omega_\phi, \frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\theta = \Omega_\theta\right]$$

```
subs( subslist, SymTop ):
```

```
solve( {op(")},
       {diff(Omega[phi],t), diff(Omega[psi],t), diff(Omega[theta],t)} ):
```

```
collect( ", [I[xy],Omega[psi],sin(psi),Omega[theta]], factor );
```

$$\left\{\frac{\partial}{\partial t}\Omega_\phi = \frac{((1-\beta)\,\Omega_\theta - \cos(\psi)\,\Omega_\phi\,(1+\beta))\,\Omega_\psi}{\sin(\psi)} + \frac{K_x \sin(\theta) + \cos(\theta)\,K_y}{\sin(\psi)\,I_{xy}},\right.$$

$$\frac{\partial}{\partial t}\Omega_\psi = \left(\beta\cos(\psi)\,\Omega_\phi^{\,2} + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \frac{-K_y\sin(\theta) + \cos(\theta)\,K_x}{I_{xy}}, \frac{\partial}{\partial t}\Omega_\theta =$$

$$\left(-\Omega_\phi\,\beta\sin(\psi) + \frac{(-1+\beta)\cos(\psi)\,\Omega_\theta + (1+\beta)\,\Omega_\phi}{\sin(\psi)}\right)\Omega_\psi$$

$$\left.+\frac{-\frac{K_z}{-1+\beta} - \frac{(K_x\sin(\theta) + \cos(\theta)\,K_y)\cos(\psi)}{\sin(\psi)}}{I_{xy}}\right\}$$

```
foo := ":
```

```
select( has, foo, diff(Omega[phi],t) ):
```

```
collect( op(")*sin(psi), [I[xy],Omega[psi],Omega[theta]], factor ):
```

```
foo := foo minus "" union {"}:
```

```
select( has, foo, diff(Omega[theta],t) ):
```

```
collect( sinfix(op(")*sin(psi),psi),
         [I[xy],Omega[psi],Omega[phi],K[z]], factor ):
```

```
foo := foo minus "" union {"};
```

$$foo := \left\{\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi\right.$$

$$+\frac{-\frac{\sin(\psi)\,K_z}{-1+\beta} - (K_x\sin(\theta) + \cos(\theta)\,K_y)\cos(\psi)}{I_{xy}},$$

$$\frac{\partial}{\partial t}\Omega_{\psi} = \left(\beta \cos(\psi)\,\Omega_{\phi}^{2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + \frac{-K_y \sin(\theta) + \cos(\theta)\,K_x}{I_{xy}},$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right) = ((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi} + \frac{K_x \sin(\theta) + \cos(\theta)\,K_y}{I_{xy}}\Bigg]$$

```
FirstOrderODEsK := [
  diff(phi,t)=Omega[phi],
  diff(psi,t)=Omega[psi],
  diff(phi,t)=Omega[phi],
  op(select(has,foo,diff(Omega[phi],t))),
  op(select(has,foo,diff(Omega[psi],t))),
  op(select(has,foo,diff(Omega[theta],t)))
];
```

$$FirstOrderODEsK := \Bigg[\frac{\partial}{\partial t}\phi = \Omega_{\phi}, \frac{\partial}{\partial t}\psi = \Omega_{\psi}, \frac{\partial}{\partial t}\phi = \Omega_{\phi},$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right) = ((1-\beta)\,\Omega_{\theta} - \cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi} + \frac{K_x \sin(\theta) + \cos(\theta)\,K_y}{I_{xy}},$$

$$\frac{\partial}{\partial t}\Omega_{\psi} = \left(\beta \cos(\psi)\,\Omega_{\phi}^{2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + \frac{-K_y \sin(\theta) + \cos(\theta)\,K_x}{I_{xy}},$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_{\phi} + (-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi}$$

$$+ \frac{-\frac{\sin(\psi)\,K_z}{-1+\beta} - (K_x \sin(\theta) + \cos(\theta)\,K_y)\cos(\psi)}{I_{xy}}\Bigg]$$

```
latex(mat(FirstOrderODEsK),`d:/dynamics/precession/FirstOrderODEsK.tex`)
;
```

Torques in the body frame

```
mat(K[x],K[y],K[z]) = r*cross( vec(0,0,1),
                               R(phi,psi,theta) &* vec(F[X],F[Y],F[Z])
);
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = r\,[-(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi))\,F_X$$

$$-(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi))\,F_Y - \cos(\theta)\sin(\psi)\,F_Z,$$

$$(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,F_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,F_Y + \sin(\theta)\sin(\psi)\,F_Z,\ 0]$$

where $[F_X, F_Y, F_Z]$ is the force vector in the fixed frame, and the effective moment arm of the force lies along the body symmetry axis (factor r).

```
cross( vec(0,0,1), R(phi,psi,theta) &* vec(F[X],F[Y],F[Z]) );
```

$$[-(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi))\,F_X - (-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi))\,F_Y - \cos(\theta)\sin(\psi)\,F_Z,\ (\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,F_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,F_Y + \sin(\theta)\sin(\psi)\,F_Z,\ 0]$$

```
collect( convert(",list), [F[X],F[Y],F[Z]], factor );
```

$$[(\sin(\theta)\cos(\phi) + \cos(\theta)\cos(\psi)\sin(\phi))\,F_X + (\sin(\theta)\sin(\phi) - \cos(\theta)\cos(\psi)\cos(\phi))\,F_Y - \cos(\theta)\sin(\psi)\,F_Z,\ (\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,F_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,F_Y + \sin(\theta)\sin(\psi)\,F_Z,\ 0]$$

```
torque := ":
```

```
mat(K[x],K[y],K[z]) = r*mat(op(torque));
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = r\,[(\sin(\theta)\cos(\phi) + \cos(\theta)\cos(\psi)\sin(\phi))\,F_X + (\sin(\theta)\sin(\phi) - \cos(\theta)\cos(\psi)\cos(\phi))\,F_Y - \cos(\theta)\sin(\psi)\,F_Z]\ [(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,F_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,F_Y + \sin(\theta)\sin(\psi)\,F_Z]\ [0]$$

```
latex(",`d:/dynamics/precession/torquesF.tex`);
```

The first-order equations of motion become

```
foo := subs( K[x]=torque[1], K[y]=torque[2], K[z]=torque[3],
FirstOrderODEsK ):
```

```
collect( foo[4], [I[xy],Omega[psi],Omega[theta]], factor ):
```

```
termfunc( ", simplify ):
```

```
collect( ", [I[xy],Omega[psi],Omega[theta]], factor );
```

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right)=((1-\beta)\,\Omega_{\theta}-\cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi}+\frac{F_X\cos(\phi)+\sin(\phi)\,F_Y}{I_{xy}}$$

```
foo[4] := ":
collect( foo[5], [I[xy],Omega[psi],Omega[phi],K[z]], factor ):
termfunc( ", simplify ):
collect( ", [I[xy],sin(psi),cos(psi)], factor );
```

$$\frac{\partial}{\partial t}\Omega_{\psi}=\left(\beta\cos(\psi)\,\Omega_{\phi}^{\;2}+(-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi)+\frac{-F_Z\sin(\psi)+(F_X\sin(\phi)-F_Y\cos(\phi))\cos(\psi)}{I_{xy}}$$

```
foo[5] := ":
collect( foo[6], [I[xy],Omega[psi],Omega[phi],K[z]], factor ):
termfunc( ", simplify ):
collect( ", [I[xy],Omega[psi],Omega[phi],K[z]], factor );
```

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right)=((\cos(\psi)^2\,\beta+1)\,\Omega_{\phi}+(-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi}-\frac{\cos(\psi)\,(F_X\cos(\phi)+\sin(\phi)\,F_Y)}{I_{xy}}$$

```
foo[6] := ":
FirstOrderODEsF := foo;
```

$$FirstOrderODEsF := \left[\frac{\partial}{\partial t}\phi=\Omega_{\phi},\frac{\partial}{\partial t}\psi=\Omega_{\psi},\frac{\partial}{\partial t}\phi=\Omega_{\phi},\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right)=((1-\beta)\,\Omega_{\theta}-\cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi}+\frac{F_X\cos(\phi)+\sin(\phi)\,F_Y}{I_{xy}},\frac{\partial}{\partial t}\Omega_{\psi}=\left(\beta\cos(\psi)\,\Omega_{\phi}^{\;2}+(-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi)+\frac{-F_Z\sin(\psi)+(F_X\sin(\phi)-F_Y\cos(\phi))\cos(\psi)}{I_{xy}},\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right)=((\cos(\psi)^2\,\beta+1)\,\Omega_{\phi}+(-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi}-\frac{\cos(\psi)\,(F_X\cos(\phi)+\sin(\phi)\,F_Y)}{I_{xy}}\right]$$

where the force components have been multiplied by the moment arm r.

```
latex(mat(FirstOrderODEsF),`d:/dynamics/precession/FirstOrderODEsF.tex`)
```

```
;
```

Force-Free Motion of a Symmetric Top

```
FFSymTop := subs( K[x]=0, K[y]=0, K[z]=0, SymTop );
```

$$FFSymTop := \left[\left(\frac{\partial^2}{\partial t^2}\psi\right)\cos(\theta)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi) + \left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\sin(\theta) + \left(-\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta), -\left(\frac{\partial^2}{\partial t^2}\psi\right)\sin(\theta) + \left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta) + \left(\cos(\psi)(1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\phi\right)+(-1+\beta)\left(\frac{\partial}{\partial t}\psi\right)\left(\frac{\partial}{\partial t}\theta\right)\right)\cos(\theta), \left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\right]$$

We notice that the third equation,

```
FFSymTop[3];
```

$$\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)$$

can be written as

```
'diff(diff(phi,t)*cos(psi)+diff(theta,t),t)' = FFSymTop[3];
```

$$\frac{\partial}{\partial t}\left(\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)\right)=\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)-\left(\frac{\partial}{\partial t}\phi\right)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)$$

```
rhs(")-lhs(");
```

$$0$$

Hence,

```
diff(phi,t)*cos(psi)+diff(theta,t) = const;
```

$$\left(\frac{\partial}{\partial t}\phi\right)\cos(\psi)+\left(\frac{\partial}{\partial t}\theta\right)=const$$

This is the projection of the angular velocity onto the symmetry axis. Recall the components of Ω in the body frame:

```
mat( Omega[x], Omega[y], Omega[z] ) = eval(Obody);
```

$$\begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial}{\partial t}\phi\right)\sin(\theta)\sin(\psi) + \left(\frac{\partial}{\partial t}\psi\right)\cos(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\theta)\sin(\psi) - \left(\frac{\partial}{\partial t}\psi\right)\sin(\theta) \\ \left(\frac{\partial}{\partial t}\phi\right)\cos(\psi) + \left(\frac{\partial}{\partial t}\theta\right) \end{bmatrix}$$

We see that the angular velocity about the symmetry axis, Ω_z, is a constant of the motion. We also have conservation of energy,

```
KE := E = 1/2*(I[x]*Omega[x]^2+I[y]*Omega[y]^2+I[z]*Omega[z]^2);
```

$$KE := E = \frac{1}{2} I_x \Omega_x^{\ 2} + \frac{1}{2} I_y \Omega_y^{\ 2} + \frac{1}{2} I_z \Omega_z^{\ 2}$$

Substituting the components of Ω in terms of the Euler angles, we have

```
KE := collect( subs( Omega[x]=Obody[1,1], Omega[y]=Obody[2,1],
                     Omega[z]=Obody[3,1], KE ), [diff,sin(psi)], factor );
```

$$\begin{aligned} KE := E = & \left(\left(\frac{1}{2} I_y \cos(\theta)^2 + \frac{1}{2} I_x \sin(\theta)^2\right)\sin(\psi)^2 + \frac{1}{2} I_z \cos(\psi)^2\right)\left(\frac{\partial}{\partial t}\phi\right)^2 \\ & + \left(I_z\left(\frac{\partial}{\partial t}\theta\right)\cos(\psi) + \sin(\theta)\cos(\theta)\,(-I_y + I_x)\sin(\psi)\left(\frac{\partial}{\partial t}\psi\right)\right)\left(\frac{\partial}{\partial t}\phi\right) + \frac{1}{2} I_z \left(\frac{\partial}{\partial t}\theta\right)^2 \\ & + \left(\frac{1}{2} I_x \cos(\theta)^2 + \frac{1}{2} I_y \sin(\theta)^2\right)\left(\frac{\partial}{\partial t}\psi\right)^2 \end{aligned}$$

```
subslist := [diff(psi,t) = Omega[psi],diff(phi,t) =
Omega[phi],diff(theta,t) = Omega[theta]];
```

$$subslist := \left[\frac{\partial}{\partial t}\psi = \Omega_\psi, \frac{\partial}{\partial t}\phi = \Omega_\phi, \frac{\partial}{\partial t}\theta = \Omega_\theta\right]$$

```
KE := subs(subslist,I[x]=I[y],I[y]=I[xy],KE);
```

$$\begin{aligned} KE := E = & \left(\left(\frac{1}{2} I_{xy} \cos(\theta)^2 + \frac{1}{2} I_{xy} \sin(\theta)^2\right)\sin(\psi)^2 + \frac{1}{2} I_z \cos(\psi)^2\right)\Omega_\phi^{\ 2} + I_z \Omega_\theta \cos(\psi)\,\Omega_\phi \\ & + \frac{1}{2} I_z \Omega_\theta^{\ 2} + \left(\frac{1}{2} I_{xy} \cos(\theta)^2 + \frac{1}{2} I_{xy} \sin(\theta)^2\right)\Omega_\psi^{\ 2} \end{aligned}$$

```
KE := collect( simplify(KE/I[xy]),
[Omega[phi],Omega[psi],Omega[theta],I[xy]], simplify );
```

$$KE := \frac{E}{I_{xy}} = \left(\frac{1}{2} - \frac{1}{2}\cos(\psi)^2 + \frac{1}{2}\frac{I_z \cos(\psi)^2}{I_{xy}}\right)\Omega_\phi^{\ 2} + \frac{I_z \Omega_\theta \cos(\psi)\,\Omega_\phi}{I_{xy}} + \frac{1}{2}\Omega_\psi^{\ 2} + \frac{1}{2}\frac{I_z \Omega_\theta^{\ 2}}{I_{xy}}$$

Steady Precession Solution

Suppose we look for a solution such that ψ is constant. Then we have

```
eval( subs( diff(psi,t)=0, FFSymTop ) );
```

$$\left[\left(\frac{\partial^2}{\partial t^2}\phi\right)\sin(\theta)\sin(\psi)+\left(-\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(1-\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\cos(\theta),\right.$$
$$\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\theta)\sin(\psi)+\left(\beta\cos(\psi)\left(\frac{\partial}{\partial t}\phi\right)^2+(-1+\beta)\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\phi\right)\right)\sin(\psi)\sin(\theta),$$
$$\left.\left(\frac{\partial^2}{\partial t^2}\theta\right)+\left(\frac{\partial^2}{\partial t^2}\phi\right)\cos(\psi)\right]$$

The first two equations can be combined to yield

```
collect( ("[1]*sin(theta) + "[2]*cos(theta))/sin(psi),
         [diff], simplify ) = 0;
```

$$\frac{\partial^2}{\partial t^2}\phi=0$$

Hence, solutions with $\psi = const$ force a constant precession, $\frac{\partial}{\partial t}\phi = const$. From the third equation, we then see that $\frac{\partial}{\partial t}\theta = const$ as well.

```
subs( diff(phi,t,t)=0, ""[1] ):
```

```
solve(",diff(phi,t)):
```

```
diff(phi,t) = "[2];
```

$$\frac{\partial}{\partial t}\phi=-\frac{\left(\frac{\partial}{\partial t}\theta\right)(-1+\beta)}{\beta\cos(\psi)}$$

```
normal( subs( beta = (I[xy]-I[z])/I[xy], " ) );
```

$$\frac{\partial}{\partial t}\phi=-\frac{\left(\frac{\partial}{\partial t}\theta\right)I_z}{(-I_{xy}+I_z)\cos(\psi)}$$

A Fast Symmetric Top with Torque Perpendicular to the Symmetry Axis

Consider the first-order equations for Ω_ϕ, Ω_ψ, Ω_θ.

```
foo := subs( F[X]=0, F[Y]=0, FirstOrderODEsF[4..6] );
```

$$foo := \left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,\Omega_\phi\,(1+\beta))\,\Omega_\psi, \right.$$

$$\frac{\partial}{\partial t}\Omega_\psi = \left(\beta\cos(\psi)\,\Omega_\phi{}^2 + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) - \frac{F_Z\sin(\psi)}{I_{xy}},$$

$$\left. \sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) = ((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi \right]$$

```
foo[1] := foo[1]/sin(psi):
foo[3] := foo[3]/sin(psi):
```

Let Ω_θ be much larger in magnitude than the angular velocities Ω_ϕ and Ω_ψ. Expanding to first order, we have

```
foo := convert( expansion( foo, [Omega[phi],Omega[psi]], 1 ), diff );
```

$$foo := \left[\frac{\partial}{\partial t}\Omega_\phi = \frac{(1-\beta)\,\Omega_\theta\,\Omega_\psi}{\sin(\psi)}, \frac{\partial}{\partial t}\Omega_\psi = -\frac{F_Z\sin(\psi)}{I_{xy}} + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\sin(\psi), \right.$$

$$\left. \frac{\partial}{\partial t}\Omega_\theta = \frac{(-1+\beta)\cos(\psi)\,\Omega_\theta\,\Omega_\psi}{\sin(\psi)} \right]$$

Differentiate these to yield second order ODEs.

```
CollectDiffs( subs(diff(psi,t)=Omega[psi],diff(foo,t)) );
```

$$\left[\frac{\partial^2}{\partial t^2}\Omega_\phi = -\frac{(-1+\beta)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right) + \left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi\right)}{\sin(\psi)} + \frac{(-1+\beta)\,\Omega_\theta\,\Omega_\psi{}^2\cos(\psi)}{\sin(\psi)^2}, \frac{\partial^2}{\partial t^2}\Omega_\psi = \right.$$

$$(-1+\beta)\sin(\psi)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\phi\right) + \Omega_\phi\left(\frac{\partial}{\partial t}\Omega_\theta\right)\right) + \frac{\cos(\psi)\,\Omega_\psi\,(-F_Z - \Omega_\theta\,\Omega_\phi\,I_{xy} + \Omega_\theta\,\Omega_\phi\,I_{xy}\,\beta)}{I_{xy}},$$

$$\frac{\partial^2}{\partial t^2}\Omega_\theta = \frac{(-1+\beta)\cos(\psi)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right) + \left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi\right)}{\sin(\psi)} - \frac{(-1+\beta)\,\Omega_\psi{}^2\,\Omega_\theta\,(\cos(\psi)^2 + \sin(\psi)^2)}{\sin(\psi)^2}$$

$$\left. \right]$$

Expand again, this time with the additional assumption that the magnitude of the force F_Z is small.

```
bar := CollectDiffs( convert(
        expansion( ", [Omega[phi],Omega[psi],F[Z]], 1 ), diff ) );
```

$$bar := \left[\frac{\partial^2}{\partial t^2}\Omega_\phi = -\frac{(-1+\beta)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right)+\left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi\right)}{\sin(\psi)}, \frac{\partial^2}{\partial t^2}\Omega_\psi = (-1+\beta)\sin(\psi)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\phi\right)+\Omega_\phi\left(\frac{\partial}{\partial t}\Omega_\theta\right)\right), \frac{\partial^2}{\partial t^2}\Omega_\theta = \frac{(-1+\beta)\cos(\psi)\left(\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right)+\left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi\right)}{\sin(\psi)} \right]$$

Substitute the first-order ODEs in the right hand sides.

```
bar := [ seq( lhs(bar[i]) = subs( op(foo), rhs(bar[i]) ), i=1..3 ) ];
```

$$bar := \left[\frac{\partial^2}{\partial t^2}\Omega_\phi = -\frac{(-1+\beta)\left(\Omega_\theta\left(-\frac{F_Z\sin(\psi)}{I_{xy}}+(-1+\beta)\Omega_\theta\Omega_\phi\sin(\psi)\right)+\frac{(-1+\beta)\cos(\psi)\Omega_\theta\Omega_\psi^{\,2}}{\sin(\psi)}\right)}{\sin(\psi)}, \frac{\partial^2}{\partial t^2}\Omega_\psi = (-1+\beta)\sin(\psi)\left(\frac{\Omega_\theta^{\,2}(1-\beta)\Omega_\psi}{\sin(\psi)}+\frac{\Omega_\phi(-1+\beta)\cos(\psi)\Omega_\theta\Omega_\psi}{\sin(\psi)}\right), \frac{\partial^2}{\partial t^2}\Omega_\theta = \frac{(-1+\beta)\cos(\psi)\left(\Omega_\theta\left(-\frac{F_Z\sin(\psi)}{I_{xy}}+(-1+\beta)\Omega_\theta\Omega_\phi\sin(\psi)\right)+\frac{(-1+\beta)\cos(\psi)\Omega_\theta\Omega_\psi^{\,2}}{\sin(\psi)}\right)}{\sin(\psi)} \right]$$

Expanding again, we find

```
bar := collect( convert( expansion( bar, [Omega[phi],Omega[psi],F[Z]], 1 ),
diff ),
                [Omega[theta],I[xy],sin(psi)], factor );
```

$$bar := \left[\frac{\partial^2}{\partial t^2}\Omega_\phi = -(-1+\beta)^2\Omega_\theta^{\,2}\Omega_\phi+\frac{(-1+\beta)\Omega_\theta F_Z}{I_{xy}}, \frac{\partial^2}{\partial t^2}\Omega_\psi = -(-1+\beta)^2\Omega_\psi\Omega_\theta^{\,2}, \frac{\partial^2}{\partial t^2}\Omega_\theta = (-1+\beta)^2\cos(\psi)\Omega_\theta^{\,2}\Omega_\phi-\frac{(-1+\beta)\cos(\psi)\Omega_\theta F_Z}{I_{xy}} \right]$$

```
bar[1] := lhs(bar[1]) = pullout( rhs(bar[1]), (-1+beta)*Omega[theta] ):
bar[3] := lhs(bar[3]) = pullout( rhs(bar[3]), cos(psi)*(-1+beta)*Omega[theta]
):
bar;
```

$$\left[\frac{\partial^2}{\partial t^2}\Omega_\phi = (-1+\beta)\,\Omega_\theta\left(-(-1+\beta)\,\Omega_\theta\,\Omega_\phi + \frac{F_Z}{I_{xy}}\right), \frac{\partial^2}{\partial t^2}\Omega_\psi = -(-1+\beta)^2\,\Omega_\psi\,\Omega_\theta{}^2,\right.$$
$$\left.\frac{\partial^2}{\partial t^2}\Omega_\theta = (-1+\beta)\cos(\psi)\,\Omega_\theta\left((-1+\beta)\,\Omega_\theta\,\Omega_\phi - \frac{F_Z}{I_{xy}}\right)\right]$$

Since Ω_θ is large, we may assume it is slowly varying. Hence, from the third equation we have $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$ and therefore

```
op( solve( subs(diff(diff(Omega[theta],t),t)=0,bar[3]), {Omega[phi]} ) );
```

$$\Omega_\phi = \frac{F_Z}{\Omega_\theta\,I_{xy}\,(-1+\beta)}$$

or,

```
simplify( ", {beta=(I[xy]-I[z])/I[xy]}, [beta] );
```

$$\Omega_\phi = -\frac{F_Z}{\Omega_\theta\,I_z}$$

Ω_ϕ is a constant. This makes the first and third equations consistent. Thus, we have a constant retrograde precession around the Z axis. Notice that the precession frequency is independent of ψ, and that it is inversely proportional to the angular momentum about the body symmetry axis. The second equation described harmonic motion for Ω_ψ,

```
lhs(bar[2]) - rhs(bar[2]) = 0;
```

$$\left(\frac{\partial^2}{\partial t^2}\Omega_\psi\right) + (-1+\beta)^2\,\Omega_\psi\,\Omega_\theta{}^2 = 0$$

with frequency

```
omega = (1-beta)*Omega[theta];
```

$$\omega = (1-\beta)\,\Omega_\theta$$

Torques due to a Pressure Applied to a Truncated Cone

Consider a truncated cone with small and large radii a and b and cone opening angle α. Let the center of mass reside on the (local) z axis, a distance h below the smaller flat surface (the top surface of the truncated cone). Then the equation for the conical surface is

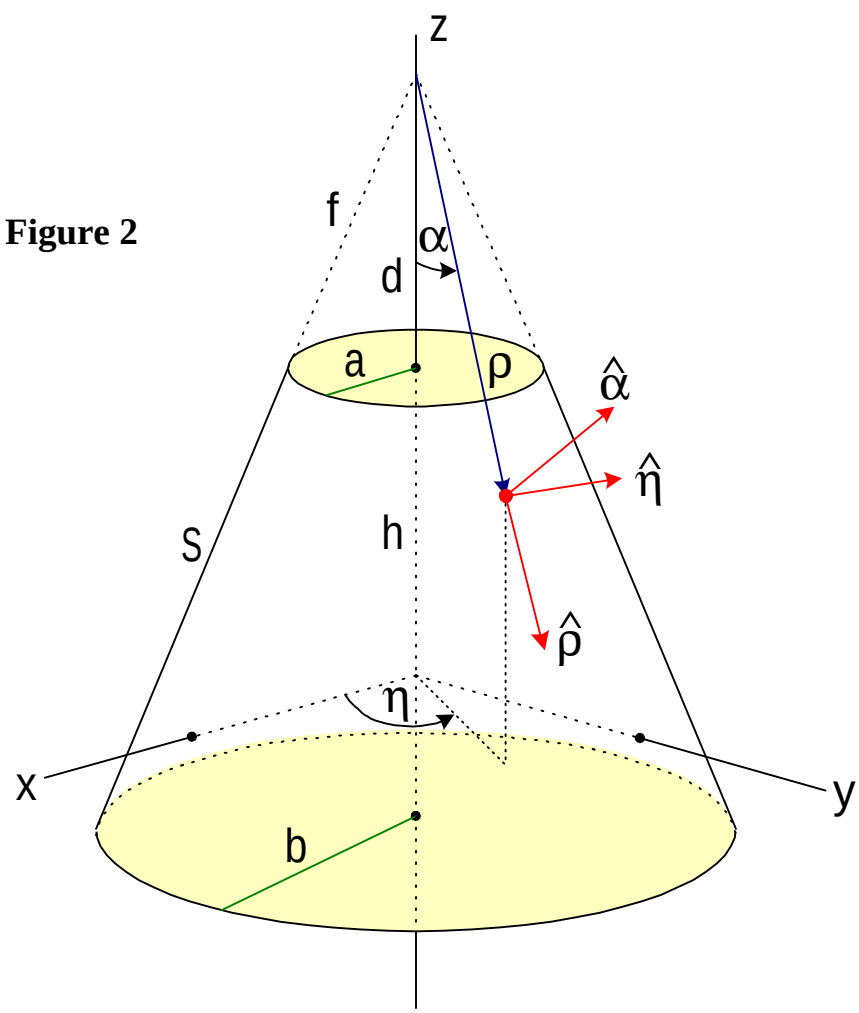
```
(x^2+y^2) - ((z-h)*tan(alpha) - a)^2 = 0;
```

$$x^2 + y^2 - ((z-h)\tan(\alpha) - a)^2 = 0$$

```
latex(",`d:/dynamics/precession/ConeEquation.tex`);
```

Conical Coordinates

Figure 2



Transformation matrix from conical to Cartesian coordinates:

```
matrix( [ [sin(alpha)*cos(eta),-sin(eta),cos(alpha)*cos(eta)],
          [sin(alpha)*sin(eta), cos(eta),cos(alpha)*sin(eta)],
          [-cos(alpha)        , 0       ,sin(alpha)] ] );
```

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix}$$

```
ConToCart := ":
```

```
latex(ConToCart,`d:/dynamics/precession/ConicalToCartesian.tex`);
```

```
M := (a,e)->evalm(subs(alpha=a,eta=e,map(simplify,inverse(ConToCart)))):
```

```
mat(xhat,yhat,zhat) = eval(ConToCart) &* mat(rhohat,etahat,alphahat);
```

$$\begin{bmatrix} xhat \\ yhat \\ zhat \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} rhohat \\ etahat \\ alphahat \end{bmatrix}$$

```
xyzhat := evalm(rhs(")):
```

Matrix to transform Cartesian partial derivatives to conical partial derivatives:

```
matrix( [ [sin(alpha)*cos(eta),      sin(alpha)*sin(eta),     -cos(alpha)
],
          [-rho*sin(alpha)*sin(eta), rho*sin(alpha)*cos(eta),  0
],
          [rho*cos(alpha)*cos(eta),  rho*cos(alpha)*sin(eta),
rho*sin(alpha) ] ] );
```

$$\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\rho\sin(\alpha)\sin(\eta) & \rho\sin(\alpha)\cos(\eta) & 0 \\ \rho\cos(\alpha)\cos(\eta) & \rho\cos(\alpha)\sin(\eta) & \rho\sin(\alpha) \end{bmatrix}$$

Invert for the inverse transformation.

```
map( collect, inverse("), [rho], simplify ):
mat(ddx,ddy,ddz) = " &* mat(ddrho,ddeta,ddalpha);
```

$$\begin{bmatrix} ddx \\ ddy \\ ddz \end{bmatrix} = \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\frac{\sin(\eta)}{\sin(\alpha)\rho} & \frac{\cos(\alpha)\cos(\eta)}{\rho} \\ \sin(\alpha)\sin(\eta) & \frac{\cos(\eta)}{\sin(\alpha)\rho} & \frac{\cos(\alpha)\sin(\eta)}{\rho} \\ -\cos(\alpha) & 0 & \frac{\sin(\alpha)}{\rho} \end{bmatrix} \&* \begin{bmatrix} ddrho \\ ddeta \\ ddalpha \end{bmatrix}$$

```
ddxyz := evalm(rhs("")):
```

Combine for gradient in conical coordinates.

```
collect( sum(ddxyz[i,1]*xyzhat[i,1],i=1..3),
         [rhohat,etahat,alphahat,ddrho,ddeta,ddalpha], simplify );
```

$$ddrho\ rhohat + \frac{ddeta\ etahat}{\rho\sin(\alpha)} + \frac{ddalpha\ alphahat}{\rho}$$

Radiation Pressure on the Cone in the Body Frame, Conical Coordinates

Let P be the pressure vector in the fixed frame and p be the pressure in the body frame. Then the conical coordinates representation of p is

```
mat(p[rho],p[eta],p[alpha]) = M(alpha,eta) &* R(phi,psi,theta) &*
mat(P[X],P[Y],P[Z]);
```

$$\begin{bmatrix} p_\rho \\ p_\eta \\ p_\alpha \end{bmatrix} = \left(\begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \right.$$

$$[\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi), \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi), \sin(\theta)\sin(\psi)]$$

$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi), -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$

$$\cos(\theta)\sin(\psi)]$$

$$\left. [\sin(\psi)\sin(\phi), -\sin(\psi)\cos(\phi), \cos(\psi)] \right) \&* \begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix}$$

```
latex(",`d:/dynamics/precession/PressureBodyFrame.tex`);
```

```
Pbody := evalm(rhs("")):
```

```
simplify(mag("));
```

$$\sqrt{P_Z^{\,2} + P_X^{\,2} + P_Y^{\,2}}$$

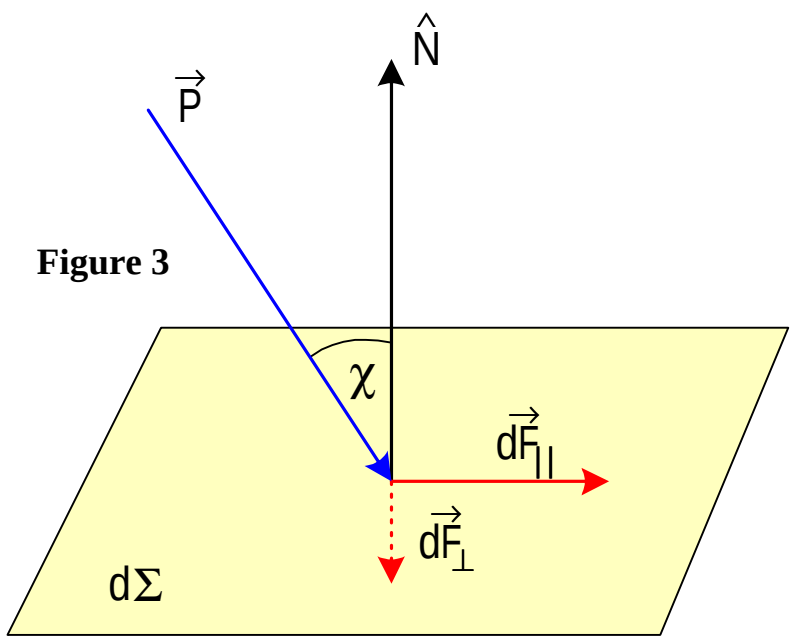


Figure 3

The force on an area element $\rho \sin\alpha \, d\rho \, d\eta$ is

```
dF = P*( -(1+A[C])*cos(chi)^2*alphahat
         + (1-A[C])*cos(chi)*(phat + cos(chi)*alphahat)
)*rho*sin(alpha)*drho*deta;
```

$$dF = P\,(-(1 + A_C)\cos(\chi)^2\, alphahat + (1 - A_C)\cos(\chi)\,(phat + \cos(\chi)\, alphahat))\,\rho\,\sin(\alpha)$$

$$drho\ deta$$

where $\cos\chi = -\frac{p_\alpha}{P}$ and A_C is the albedo of the cone surface. But

```
-(1+A[C])*cos(chi)^2*alphahat
         + (1-A[C])*cos(chi)*(phat + cos(chi)*alphahat):
```

```
" = collect(",alphahat,factor);
```

$$-(1+A_C)\cos(\chi)^2\, alphahat + (1-A_C)\cos(\chi)\,(phat + \cos(\chi)\, alphahat) =$$

$$-2\cos(\chi)^2 A_C\, alphahat - (-1+A_C)\cos(\chi)\, phat$$

```
cos_chi := subs( P[X]=pi[X],P[Y]=pi[Y],P[Z]=pi[Z], -Pbody[3,1] );
```

$$cos_chi := -(\cos(\alpha)\cos(\eta)(\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi)) + \sin(\alpha)\sin(\psi)\sin(\phi))\,\pi_X - (\cos(\alpha)\cos(\eta)(\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi)) + \cos(\alpha)\sin(\eta)(-\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi)) - \sin(\alpha)\sin(\psi)\cos(\phi))\,\pi_Y - (\cos(\alpha)\cos(\eta)\sin(\theta)\sin(\psi) + \cos(\alpha)\sin(\eta)\cos(\theta)\sin(\psi) + \sin(\alpha)\cos(\psi))\,\pi_Z$$

```
eval(subs(alpha=Pi/2,cos_chi));
```

$$-\pi_X\sin(\phi)\sin(\psi) - \pi_Z\cos(\psi) + \pi_Y\sin(\psi)\cos(\phi)$$

Values of η for which $\cos\chi$ is an Extremum

```
collect(diff(cos_chi,eta),[sin(eta),cos(eta)]):
```

```
eta[0] = solve(",eta);
```

$$\eta_0 = -\arctan\left(\left(\pi_X\sin(\theta)\cos(\phi) + \cos(\psi)\,\pi_X\cos(\theta)\sin(\phi) + \pi_Y\sin(\theta)\sin(\phi) - \cos(\psi)\,\pi_Y\cos(\theta)\cos(\phi) - \cos(\theta)\sin(\psi)\,\pi_Z\right) \Big/ \left(-\cos(\psi)\,\pi_X\sin(\theta)\sin(\phi) + \pi_X\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\psi)\,\pi_Z + \pi_Y\cos(\theta)\sin(\phi) + \cos(\psi)\,\pi_Y\sin(\theta)\cos(\phi)\right)\right)$$

```
foobar := ":
```

```
select(has,rhs(foobar),pi[X]):
op(1,"):
location(foobar,"):
collect("",[pi[X],pi[Y],pi[Z]]):
subsop( ""=", foobar );
foobar := ":
```

$$\eta_0 = -\arctan\left(\left((\cos(\theta)\cos(\psi)\sin(\phi) + \sin(\theta)\cos(\phi))\,\pi_X + (\sin(\theta)\sin(\phi) - \cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y - \cos(\theta)\sin(\psi)\,\pi_Z\right) \Big/ \left((\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X + (\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y\right.$$

$$+ \sin(\theta)\sin(\psi)\pi_Z)$$

```
eval(subs(theta=0,foobar));
```

$$\eta_0 = -\arctan\left(\frac{\pi_X \cos(\psi)\sin(\phi) - \sin(\psi)\pi_Z - \cos(\psi)\pi_Y \cos(\phi)}{\sin(\phi)\pi_Y + \pi_X \cos(\phi)}\right)$$

```
latex(cos_chi,`d:/dynamics/precession/cos_chi.tex`);
```

Hence,

```
Fintegrand := ( (1-A[C])*cos(chi)*mat(p[rho],p[eta],p[alpha])
                - 2*P*A[C]*cos(chi)^2*mat(0,0,1) )*rho*sin(alpha);
```

$$Fintegrand := \left((1 - A_C)\cos(\chi)\begin{bmatrix} p_\rho \\ p_\eta \\ p_\alpha \end{bmatrix} - 2\,P\,A_C \cos(\chi)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\rho \sin(\alpha)$$

```
mat(F[rho],F[eta],F[alpha]) = Int( Int( Fintegrand, rho=f..f+S ),
eta=0..2*Pi );
```

$$\begin{bmatrix} F_\rho \\ F_\eta \\ F_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \left((1 - A_C)\cos(\chi)\begin{bmatrix} p_\rho \\ p_\eta \\ p_\alpha \end{bmatrix} - 2\,P\,A_C \cos(\chi)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\rho \sin(\alpha)\, d\rho\, d\eta$$

```
latex(",`d:/dynamics/precession/ConeForceIntegral.tex`);
```

Torque Due to Radiation Pressure on the Cone Surface

The torque is

```
mat(K[rho],K[eta],K[alpha]) =
    Int( Int( 'cross'(mat(x,y,z),Fintegrand), rho=f..f+S ), eta=0..2*Pi );
```

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi}\int_f^{f+S} \mathrm{cross}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \left((1 - A_C)\cos(\chi)\begin{bmatrix} p_\rho \\ p_\eta \\ p_\alpha \end{bmatrix} - 2\,P\,A_C \cos(\chi)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\rho \sin(\alpha)\right) d\rho\, d\eta$$

The distance vector from the center of mass in conical coordinates is

```
r_cone := M(alpha,eta) &* mat(x,y,z);
```

$$r_cone := \begin{bmatrix} \sin(\alpha)\cos(\eta) & \sin(\alpha)\sin(\eta) & -\cos(\alpha) \\ -\sin(\eta) & \cos(\eta) & 0 \\ \cos(\alpha)\cos(\eta) & \cos(\alpha)\sin(\eta) & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

```
xyz := [ x=rho*sin(alpha)*cos(eta), y=rho*sin(alpha)*sin(eta),
z=h+a/tan(alpha)-rho*cos(alpha) ];
```

$$xyz := \left[x = \rho\sin(\alpha)\cos(\eta), y = \rho\sin(\alpha)\sin(\eta), z = h + \frac{a}{\tan(\alpha)} - \rho\cos(\alpha) \right]$$

```
r_cone := map( collect, evalm( subs( xyz, r_cone ) ), [h,a], simplify );
```

$$r_cone := \begin{bmatrix} -\cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)} + \rho \\ 0 \\ \sin(\alpha)\,h + \cos(\alpha)\,a \end{bmatrix}$$

```
latex(mat(xyz),`d:/dynamics/precession/ConicalCoordinates.tex`);
latex(r_cone,`d:/dynamics/precession/r_cone.tex`);
```

Hence, the torque becomes

```
Kintegrand := mat(cross(r_cone,Fintegrand)):
mat(K[rho],K[eta],K[alpha]) =
    Int( Int( eval(Kintegrand), rho=f..f+S ), eta=0..2*Pi );
```

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \int_f^{f+S} \left[-(\sin(\alpha)\,h + \cos(\alpha)\,a)\,\rho\sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\eta \right] \left[(\sin(\alpha)\,h + \cos(\alpha)\,a)\,\rho\sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\rho - \left(-\cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)} + \rho \right) \rho\sin(\alpha)\,((1 - A_C)\cos(\chi)\,p_\alpha - 2\,P\,A_C\cos(\chi)^2) \right] \left[\left(-\cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)} + \rho \right) \rho\sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\eta \right] d\rho\,d\eta$$

Since the pressure components are independent of ρ, the integral over ρ can be done right away.

```
mat(K[rho],K[eta],K[alpha]) =
    Int( subs( p[r]=p[rho],

mat(map(int,subs(p[rho]=p[r],convert(Kintegrand,vector)),rho)) ),
         eta=0..2*pi );
```

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} -\frac{1}{2}(\sin(\alpha)\,h + \cos(\alpha)\,a)\,\rho^2 \sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\eta \\ \frac{1}{2}(\sin(\alpha)\,h + \cos(\alpha)\,a)\,\rho^2 \sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\rho \\ - \sin(\alpha)\,((1 - A_C)\cos(\chi)\,p_\alpha - 2\,P\,A_C \cos(\chi)^2)\left(\frac{1}{3}\rho^3 + \frac{1}{2}\left(-\cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)\rho^2\right) \\ \sin(\alpha)\,(1 - A_C)\cos(\chi)\,p_\eta\left(\frac{1}{3}\rho^3 + \frac{1}{2}\left(-\cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)\rho^2\right) \end{bmatrix} d\eta$$

```
torque_cone := ":
```

Now we will substitute for $\cos\xi$ and p_ρ, p_η, p_α. First, we make the substitutions

```
[ B[1] = 1/2*(sin(alpha)*h+cos(alpha)*a)*rho^2*sin(alpha)*(1-A[C]),
  B[2] = (1/2*(2/3*rho+(-cos(alpha)*h-cos(alpha)^2/sin(alpha)*a))*rho^2)
         * sin(alpha) ];
```

$$\left[B_1 = \frac{1}{2}(\sin(\alpha)\,h + \cos(\alpha)\,a)\,\rho^2 \sin(\alpha)\,(1 - A_C),\quad B_2 = \frac{1}{2}\left(\frac{2}{3}\rho - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)\rho^2 \sin(\alpha)\right]$$

```
torque_coeffs := ":
```

so that

```
mat(K[rho],K[eta],K[alpha]) =
    Int( mat( -B[1]*cos(chi)*p[eta],

B[1]*cos(chi)*p[rho]-B[2]*((1-A[C])*cos(chi)*p[alpha]-2*P*A[C]*cos(chi)^2),

                B[2]*(1-A[C])*cos(chi)*p[eta] ), eta=0..2*Pi );
```

$$\begin{bmatrix} K_\rho \\ K_\eta \\ K_\alpha \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} -B_1 \cos(\chi)\,p_\eta \\ B_1 \cos(\chi)\,p_\rho - B_2\,((1 - A_C)\cos(\chi)\,p_\alpha - 2\,P\,A_C \cos(\chi)^2) \\ B_2\,(1 - A_C)\cos(\chi)\,p_\eta \end{bmatrix} d\eta$$

```
tmp := ":
```

Check:

```
subs( torque_coeffs, op([2,1],") ):
```

```
foobar := ":
seq( simplify( foobar[i,1] - op([2,1],torque_cone)[i,1] ), i=1..3 );
```

$$0, 0, 0$$

```
torque_cone := tmp:
latex(torque_cone,`d:/dynamics/precession/ConeTorqueIntegralCone.tex`);
```

Do the integral limits for ρ now.

```
subs( rho^2=(b^2-a^2)/sin(alpha)^2, rho=(b-a)/sin(alpha), torque_coeffs );
```

$$\left[B_1 = \frac{1}{2}\frac{(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2 - a^2)\,(1 - A_C)}{\sin(\alpha)},\ B_2 = \frac{1}{2}\frac{\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2 - a^2)}{\sin(\alpha)}\right]$$

```
torque_coeffs := ":
latex(torque_coeffs,`d:/dynamics/precession/torque_coeffs.tex`);
```

Now transform back to the body [x, y, z] frame.

```
mat(K[x],K[y],K[z]) =
  Int( eval(ConToCart) &* mat(dK[rho],dK[eta],dK[alpha]), eta=0..2*Pi );
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \begin{bmatrix} \sin(\alpha)\cos(\eta) & -\sin(\eta) & \cos(\alpha)\cos(\eta) \\ \sin(\alpha)\sin(\eta) & \cos(\eta) & \cos(\alpha)\sin(\eta) \\ -\cos(\alpha) & 0 & \sin(\alpha) \end{bmatrix} \&* \begin{bmatrix} dK_\rho \\ dK_\eta \\ dK_\alpha \end{bmatrix} d\eta$$

```
mat(K[x],K[y],K[z]) =
  Int( evalm(ConToCart &* op([2,1],torque_cone)), eta=0..2*Pi );
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} [-\sin(\alpha)\cos(\eta)\,B_1\cos(\chi)\,p_\eta - \sin(\eta)\,(B_1\cos(\chi)\,p_\rho - B_2\,((1 - A_C)\cos(\chi)\,p_\alpha - 2\,P\,A_C\cos(\chi)^2)) + \cos(\alpha)\cos(\eta)\,B_2\,(1 - A_C)\cos(\chi)\,p_\eta]$$

$$[-\sin(\alpha)\sin(\eta)\,B_1\cos(\chi)\,p_\eta + \cos(\eta)\,(B_1\cos(\chi)\,p_\rho - B_2\,((1 - A_C)\cos(\chi)\,p_\alpha - 2\,P\,A_C\cos(\chi)^2))$$

$$+\cos(\alpha)\sin(\eta)\,B_2\,(1-A_C)\cos(\chi)\,p_\eta\,]$$

$$[\cos(\alpha)\,B_1\cos(\chi)\,p_\eta+\sin(\alpha)\,B_2\,(1-A_C)\cos(\chi)\,p_\eta\,]\,d\eta$$

```
map( collect, op([2,1],"), [p[eta],sin(eta),cos(eta),B[2]], factor ):
subsop( [2,1]=", "" );
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi}$$

$$[(-\cos(\alpha)\,(-1+A_C)\cos(\chi)\,B_2-\sin(\alpha)\,B_1\cos(\chi))\cos(\eta)\,p_\eta$$

$$+(-\cos(\chi)\,(-p_\alpha+p_\alpha A_C+2\cos(\chi)\,P\,A_C)\,B_2-B_1\cos(\chi)\,p_\rho)\sin(\eta)]$$

$$[(-\cos(\alpha)\,(-1+A_C)\cos(\chi)\,B_2-\sin(\alpha)\,B_1\cos(\chi))\sin(\eta)\,p_\eta$$

$$+(\cos(\chi)\,(-p_\alpha+p_\alpha A_C+2\cos(\chi)\,P\,A_C)\,B_2+B_1\cos(\chi)\,p_\rho)\cos(\eta)]$$

$$[(-\sin(\alpha)\,(-1+A_C)\cos(\chi)\,B_2+\cos(\alpha)\,B_1\cos(\chi))\,p_\eta\,]\,d\eta$$

```
torque_integral_xyz := ":
```

```
subslist := [ cos(chi)=cos_chi, p[rho]=Pbody[1,1], p[eta]=Pbody[2,1],
              p[alpha]=Pbody[3,1] ]:
```

```
subs( subslist, op([2,1],torque_integral_xyz) ):
subs( P[X]=pi[X], P[Y]=pi[Y], P[Z]=pi[Z], " ):
subs( P=1, " ):
map( int, ", eta ):
convert(convert(",vector),list):
foo := ":
for ii from 1 to 3 do
  foo[ii] := factor( eval( subs(eta=2*Pi,foo[ii]) - subs(eta=0,foo[ii]) )
):
od:
mat(K[x],K[y],K[z]) = Pi*P*mat(convert(subs(Pi=1,foo),vector));
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \pi\,P$$

$$[-(\pi_X\sin(\phi)\sin(\psi)-\pi_Y\sin(\psi)\cos(\phi)+\pi_Z\cos(\psi))\,(\cos(\theta)\sin(\psi)\,\pi_Z$$

$$-\cos(\psi)\,\pi_X\cos(\theta)\sin(\phi)-\pi_X\sin(\theta)\cos(\phi)-\pi_Y\sin(\theta)\sin(\phi)+\cos(\psi)\,\pi_Y\cos(\theta)\cos(\phi))$$

$$(\cos(\alpha)\,B_2\,A_C\sin(\alpha)+B_1\cos(\alpha)^2+3\,B_2\cos(\alpha)\sin(\alpha)-2\,B_1\sin(\alpha)^2)]$$

$$[(\pi_X\sin(\phi)\sin(\psi)-\pi_Y\sin(\psi)\cos(\phi)+\pi_Z\cos(\psi))\,(-\cos(\psi)\,\pi_X\sin(\theta)\sin(\phi)$$

$$+\pi_X\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\psi)\,\pi_Z+\pi_Y\cos(\theta)\sin(\phi)+\cos(\psi)\,\pi_Y\sin(\theta)\cos(\phi))$$

$$(\cos(\alpha)\,B_2\,A_C\sin(\alpha)+B_1\cos(\alpha)^2+3\,B_2\cos(\alpha)\sin(\alpha)-2\,B_1\sin(\alpha)^2)]$$

```
    [0]
torque_xyz := ":
latex(torque_xyz,`d:/dynamics/precession/ConeTorqueBody.tex`);
```

Torque Due to Radiation Pressure on the Flattop Surface

The force on an area element $r\,drd\lambda$ is

```
dF = P*( -(1+A[T])*cos(chi)^2*zhat
         + (1-A[T])*cos(chi)*(phat + cos(chi)*zhat) )*r*dr*dlambda;
```

$$dF = P\,(-(1+A_T)\cos(\chi)^2\,zhat + (1-A_T)\cos(\chi)\,(phat + \cos(\chi)\,zhat))\,r\,dr\,dlambda$$

where $\cos\chi = -\frac{p_z}{P}$ and A_T is the albedo of the flattop. But

```
-(1+A[T])*cos(chi)^2*zhat
         + (1-A[T])*cos(chi)*(phat + cos(chi)*zhat):
" = collect(",zhat,factor);
```

$$-(1+A_T)\cos(\chi)^2\,zhat + (1-A_T)\cos(\chi)\,(phat + \cos(\chi)\,zhat) =$$

$$-2\cos(\chi)^2\,A_T\,zhat - (-1+A_T)\cos(\chi)\,phat$$

Now,

```
mat(P[x],P[y],P[z]) = R(phi,psi,theta) &* mat(P[X],P[Y],P[Z]);
```

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} =$$

$$[\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\psi)\sin(\phi),\ \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\psi)\cos(\phi),\ \sin(\theta)\sin(\psi)]$$
$$[-\sin(\theta)\cos(\phi) - \cos(\theta)\cos(\psi)\sin(\phi),\ -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\psi)\cos(\phi),$$
$$\cos(\theta)\sin(\psi)]$$

$$[\sin(\psi)\sin(\phi),\ -\sin(\psi)\cos(\phi),\ \cos(\psi)]\ \&*\begin{bmatrix} P_X \\ P_Y \\ P_Z \end{bmatrix}$$

```
latex(",`d:/dynamics/precession/FlatTopPressureBodyFrame.tex`);
PbodyT := evalm(rhs(")):
```

Hence,

```
cos_chiT := subs( P[X]=pi[X],P[Y]=pi[Y],P[Z]=pi[Z], -PbodyT[3,1] ):
cos(chi) = ";
```

$$\cos(\chi) = -\pi_X\sin(\phi)\sin(\psi) - \pi_Z\cos(\psi) + \pi_Y\sin(\psi)\cos(\phi)$$

The angle χ for the cone surface when $\alpha = \frac{\pi}{2}$ should match this:

```
eval(subs(alpha=Pi/2,cos_chi));
```

$$-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi)$$

```
latex(cos_chiT,`d:/dynamics/precession/cos_chiFlatTop.tex`);
```

The force integral is then

```
( (1-A[T])*cos(chi)*mat(P[x],P[y],P[z])
                 - 2*P*A[T]*cos(chi)^2*mat(0,0,1) )*r:
mat(F[x],F[y],F[z]) = Int( Int( ", r=0..a ), lambda=0..2*Pi );
```

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \int_0^{2\pi}\int_0^{a} \left((1 - A_T)\cos(\chi) \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} - 2\,P\,A_T \cos(\chi)^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) r\,dr\,d\lambda$$

```
latex(",`d:/dynamics/precession/FlatTopForceIntegral.tex`);
```

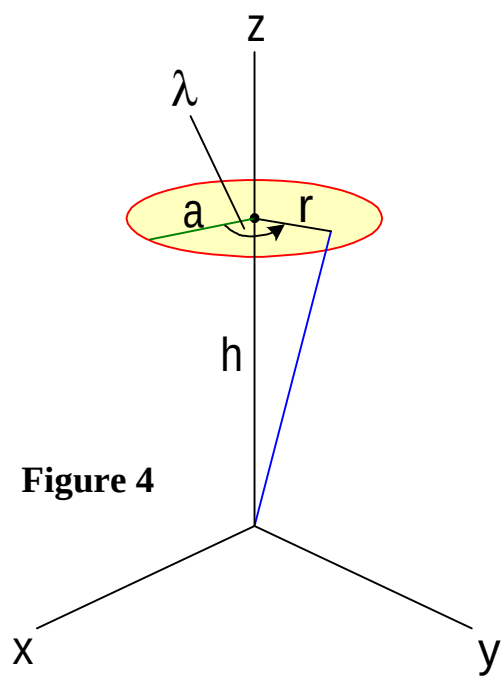


Figure 4

The torque due to pressure on the flattop is

```
mat(K[x],K[y],K[z]) =
    Int( Int( mat(cross( mat(r*cos(lambda),r*sin(lambda),h),
evalm(op([2,1,1],"))) ),
    r=0..a ), lambda=0..2*Pi );
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \int_0^{2\pi} \int_0^{a} \begin{bmatrix} r^2 \sin(\lambda)\left((1-A_T)\cos(\chi) P_z - 2 P A_T \cos(\chi)^2\right) - h\, r\,(1-A_T)\cos(\chi) P_y \\ h\, r\,(1-A_T)\cos(\chi) P_x - r^2 \cos(\lambda)\left((1-A_T)\cos(\chi) P_z - 2 P A_T \cos(\chi)^2\right) \\ r^2 \cos(\lambda)(1-A_T)\cos(\chi) P_y - r^2 \sin(\lambda)(1-A_T)\cos(\chi) P_x \end{bmatrix} dr\, d\lambda$$

```
latex(",`d:/dynamics/precession/FlatTopTorqueIntegral.tex`);
```

Evaluating the integrals, we have

```
map( int, op([2,1,1],"), r=0..a );
map( int, ", lambda=0..2*Pi );
map(factor,");
mat(K[x],K[y],K[z]) = P*Pi*a^2*h*(1-A[T])*cos(chi)*
                      subs( P[X]=pi[X], P[Y]=pi[Y], P[Z]=pi[Z],
                            a=1, h=1, Pi=1, cos(chi)=1, A[T]=0,
                            subs( P[x]=PbodyT[1,1], P[y]=PbodyT[2,1],
                                  map( factor, " ) ) ):
subs( cos(chi)=cos_chiT, " );
```

$$\left[\frac{1}{3} a^3 \sin(\lambda)\cos(\chi) P_z - \frac{1}{3} a^3 \sin(\lambda)\cos(\chi) P_z A_T - \frac{2}{3} a^3 \sin(\lambda) P A_T \cos(\chi)^2 - \frac{1}{2} h\, a^2 \cos(\chi) P_y + \frac{1}{2} h\, a^2 \cos(\chi) P_y A_T\right]$$

$$\left[\frac{1}{2} h\, a^2 \cos(\chi) P_x - \frac{1}{2} h\, a^2 \cos(\chi) P_x A_T - \frac{1}{3} a^3 \cos(\lambda)\cos(\chi) P_z + \frac{1}{3} a^3 \cos(\lambda)\cos(\chi) P_z A_T + \frac{2}{3} a^3 \cos(\lambda) P A_T \cos(\chi)^2\right]$$

$$\left[\frac{1}{3} a^3 \cos(\lambda)\cos(\chi) P_y - \frac{1}{3} a^3 \cos(\lambda)\cos(\chi) P_y A_T - \frac{1}{3} a^3 \sin(\lambda)\cos(\chi) P_x + \frac{1}{3} a^3 \sin(\lambda)\cos(\chi) P_x A_T\right]$$

$$\begin{bmatrix} -h\, a^2 \cos(\chi) P_y \pi + h\, a^2 \cos(\chi) P_y A_T \pi \\ h\, a^2 \cos(\chi) P_x \pi - h\, a^2 \cos(\chi) P_x A_T \pi \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} P_y a^2 \cos(\chi) h \pi (-1 + A_T) \\ -h a^2 \cos(\chi) P_x \pi (-1 + A_T) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = P \pi a^2 h (1 - A_T) (-\pi_X \sin(\phi) \sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi) \cos(\phi))$$

$$[-(-\sin(\theta) \cos(\phi) - \cos(\theta) \cos(\psi) \sin(\phi)) \pi_X - (-\sin(\theta) \sin(\phi) + \cos(\theta) \cos(\psi) \cos(\phi)) \pi_Y - \cos(\theta) \sin(\psi) \pi_Z]$$

$$[(\cos(\theta) \cos(\phi) - \sin(\theta) \cos(\psi) \sin(\phi)) \pi_X + (\cos(\theta) \sin(\phi) + \sin(\theta) \cos(\psi) \cos(\phi)) \pi_Y + \sin(\theta) \sin(\psi) \pi_Z]$$

$$[0]$$

```
torque_xyzT := ":
latex(torque_xyzT,`d:/dynamics/precession/FlatTopTorqueBody.tex`);
```

Putting it All Together: the Equations of Motion

```
FirstOrderODEsK[4..6];
```

$$\left[\sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\phi \right) = ((1 - \beta) \Omega_\theta - \cos(\psi) \Omega_\phi (1 + \beta)) \Omega_\psi + \frac{K_x \sin(\theta) + \cos(\theta) K_y}{I_{xy}}, \right.$$

$$\frac{\partial}{\partial t} \Omega_\psi = \left(\beta \cos(\psi) \Omega_\phi^{\,2} + (-1 + \beta) \Omega_\theta \Omega_\phi \right) \sin(\psi) + \frac{-K_y \sin(\theta) + \cos(\theta) K_x}{I_{xy}}, \sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\theta \right)$$

$$= ((\cos(\psi)^2 \beta + 1) \Omega_\phi + (-1 + \beta) \cos(\psi) \Omega_\theta) \Omega_\psi$$

$$\left. + \frac{-\frac{\sin(\psi) K_z}{-1 + \beta} - (K_x \sin(\theta) + \cos(\theta) K_y) \cos(\psi)}{I_{xy}} \right]$$

```
torque_xyzT;  torque_xyz;  torque_coeffs;
```

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = P \pi a^2 h (1 - A_T) (-\pi_X \sin(\phi) \sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi) \cos(\phi))$$

$$[-(-\sin(\theta) \cos(\phi) - \cos(\theta) \cos(\psi) \sin(\phi)) \pi_X - (-\sin(\theta) \sin(\phi) + \cos(\theta) \cos(\psi) \cos(\phi)) \pi_Y$$

$$- \cos(\theta) \sin(\psi) \pi_Z]$$

$$[(\cos(\theta) \cos(\phi) - \sin(\theta) \cos(\psi) \sin(\phi)) \pi_X + (\cos(\theta) \sin(\phi) + \sin(\theta) \cos(\psi) \cos(\phi)) \pi_Y$$

$$+ \sin(\theta) \sin(\psi) \pi_Z]$$

$$[0]$$

$$\begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} = \pi P$$

$$[-(\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) (\cos(\theta) \sin(\psi) \pi_Z$$

$$- \cos(\psi) \pi_X \cos(\theta) \sin(\phi) - \pi_X \sin(\theta) \cos(\phi) - \pi_Y \sin(\theta) \sin(\phi) + \cos(\psi) \pi_Y \cos(\theta) \cos(\phi))$$

$$(\cos(\alpha) B_2 A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3 B_2 \cos(\alpha) \sin(\alpha) - 2 B_1 \sin(\alpha)^2)]$$

$$[(\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) (-\cos(\psi) \pi_X \sin(\theta) \sin(\phi)$$

$$+ \pi_X \cos(\theta) \cos(\phi) + \sin(\theta) \sin(\psi) \pi_Z + \pi_Y \cos(\theta) \sin(\phi) + \cos(\psi) \pi_Y \sin(\theta) \cos(\phi))$$

$$(\cos(\alpha) B_2 A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3 B_2 \cos(\alpha) \sin(\alpha) - 2 B_1 \sin(\alpha)^2)]$$

$$[0]$$

$$\left[B_1 = \frac{1}{2} \frac{(\sin(\alpha) h + \cos(\alpha) a) (b^2 - a^2) (1 - A_C)}{\sin(\alpha)}, \right.$$

$$\left. B_2 = \frac{1}{2} \frac{\left(\frac{2}{3} \frac{b - a}{\sin(\alpha)} - \cos(\alpha) h - \frac{\cos(\alpha)^2 a}{\sin(\alpha)} \right) (b^2 - a^2)}{\sin(\alpha)} \right]$$

Cone Surface Torque

Deal first with the cone surface torque.

```
foo := convert( rhs(torque_xyz)/Pi/P, vector );
```

$$foo := [-(\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) (\cos(\theta) \sin(\psi) \pi_Z$$

$$- \cos(\psi) \pi_X \cos(\theta) \sin(\phi) - \pi_X \sin(\theta) \cos(\phi) - \pi_Y \sin(\theta) \sin(\phi)$$

$$+ \cos(\psi) \pi_Y \cos(\theta) \cos(\phi))$$

$$(\cos(\alpha) B_2 A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3 B_2 \cos(\alpha) \sin(\alpha) - 2 B_1 \sin(\alpha)^2),$$

$$(\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) (-\cos(\psi) \pi_X \sin(\theta) \sin(\phi)$$

$$+ \pi_X \cos(\theta) \cos(\phi) + \sin(\theta) \sin(\psi) \pi_Z + \pi_Y \cos(\theta) \sin(\phi) + \cos(\psi) \pi_Y \sin(\theta) \cos(\phi))$$

$$(\cos(\alpha)\, B_2\, A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3\, B_2 \cos(\alpha) \sin(\alpha) - 2\, B_1 \sin(\alpha)^2),\ 0]$$

```
foo[1]*sin(theta) + foo[2]*cos(theta):
subs( sin(theta)^2+cos(theta)^2=1, factor(") );
```

$$(\cos(\alpha)\, B_2\, A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3\, B_2 \cos(\alpha) \sin(\alpha) - 2\, B_1 \sin(\alpha)^2)$$
$$(\sin(\phi)\, \pi_Y + \pi_X \cos(\phi))\, (\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi))$$

```
select( has, ", B[1] ):
loc := location("","");
applyop( collect, loc, """, [B[1],B[2]], factor ):
K[x]*sin(theta) + K[y]*cos(theta) = Pi*P*subs( torque_coeffs, " );
```

$$loc := [1]$$

$$K_x \sin(\theta) + \cos(\theta)\, K_y = \pi\, P \Bigg(\frac{1}{2} \frac{(\cos(\alpha)^2 - 2 \sin(\alpha)^2)\, (\sin(\alpha)\, h + \cos(\alpha)\, a)\, (b^2 - a^2)\, (1 - A_C)}{\sin(\alpha)} + \frac{1}{2} \cos(\alpha)\, (A_C + 3) \left(\frac{2}{3} \frac{b - a}{\sin(\alpha)} - \cos(\alpha)\, h - \frac{\cos(\alpha)^2\, a}{\sin(\alpha)} \right) (b^2 - a^2) \Bigg)$$
$$(\sin(\phi)\, \pi_Y + \pi_X \cos(\phi))\, (\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi))$$

```
tmp1 := ":
```

```
foo[1]*cos(theta) - foo[2]*sin(theta):
subs( sin(theta)^2+cos(theta)^2=1, factor(") );
```

$$(\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi))$$
$$(\pi_X \cos(\psi) \sin(\phi) - \sin(\psi)\, \pi_Z - \cos(\psi)\, \pi_Y \cos(\phi))$$
$$(\cos(\alpha)\, B_2\, A_C \sin(\alpha) + B_1 \cos(\alpha)^2 + 3\, B_2 \cos(\alpha) \sin(\alpha) - 2\, B_1 \sin(\alpha)^2)$$

```
select( has, ", B[1] ):
loc := location("","");
applyop( collect, loc, """, [B[1],B[2]], factor ):
K[x]*cos(theta) - K[y]*sin(theta) = Pi*P*subs( torque_coeffs, " );
```

$$loc := [3]$$

$$-K_y \sin(\theta) + \cos(\theta)\, K_x = \pi\, P\, (\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi))$$
$$(\pi_X \cos(\psi) \sin(\phi) - \sin(\psi)\, \pi_Z - \cos(\psi)\, \pi_Y \cos(\phi)) \Bigg($$

$$\frac{1}{2}\frac{(\cos(\alpha)^2-2\sin(\alpha)^2)(\sin(\alpha)h+\cos(\alpha)a)(b^2-a^2)(1-A_C)}{\sin(\alpha)}$$

$$+\frac{1}{2}\cos(\alpha)(A_C+3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)}-\cos(\alpha)h-\frac{\cos(\alpha)^2 a}{\sin(\alpha)}\right)(b^2-a^2)\Bigg)$$

```
tmp2 := ":
```

Flattop Torque

Now the flattop torque.

```
factr := P*Pi*a^2*h*(1-A[T]) *
(-pi[X]*sin(phi)*sin(psi)+pi[Y]*sin(psi)*cos(phi)-pi[Z]*cos(psi));
```

$$factr := P\,\pi\,a^2\,h\,(1-A_T)\,(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))$$

```
foo := convert( rhs(torque_xyzT)/factr, vector );
```

$$foo := [-(-\sin(\theta)\cos(\phi)-\cos(\theta)\cos(\psi)\sin(\phi))\,\pi_X$$
$$-(-\sin(\theta)\sin(\phi)+\cos(\theta)\cos(\psi)\cos(\phi))\,\pi_Y-\cos(\theta)\sin(\psi)\,\pi_Z,$$
$$(\cos(\theta)\cos(\phi)-\sin(\theta)\cos(\psi)\sin(\phi))\,\pi_X+(\cos(\theta)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi))\,\pi_Y$$
$$+\sin(\theta)\sin(\psi)\,\pi_Z,\ 0]$$

```
foo[1]*sin(theta) + foo[2]*cos(theta):
subs( sin(theta)^2+cos(theta)^2=1, factor(") ):
K[x]*sin(theta) + K[y]*cos(theta) = factr * ";
```

$$K_x\sin(\theta)+\cos(\theta)\,K_y=P\,\pi\,a^2\,h\,(1-A_T)$$
$$(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))\,(\sin(\phi)\,\pi_Y+\pi_X\cos(\phi))$$

```
tmp1T := ":
```

```
foo[1]*cos(theta) - foo[2]*sin(theta):
subs( sin(theta)^2+cos(theta)^2=1, factor(") ):
K[x]*cos(theta) - K[y]*sin(theta) = factr * ";
```

$$-K_y\sin(\theta)+\cos(\theta)\,K_x=P\,\pi\,a^2\,h\,(1-A_T)$$
$$(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))$$
$$(\pi_X\cos(\psi)\sin(\phi)-\sin(\psi)\,\pi_Z-\cos(\psi)\,\pi_Y\cos(\phi))$$

```
tmp2T := ":
```

Equations of Motion

```
FirstOrderODEsP := copy(FirstOrderODEsK):
select( has, op(2,FirstOrderODEsP[4]), K[x] );
```

```
location(FirstOrderODEsP[4],"):
FirstOrderODEsP[4] := subsop( "=rhs(tmp1)/I[xy]+rhs(tmp1T)/I[xy],
FirstOrderODEsP[4] );
```

$$\frac{K_x \sin(\theta) + \cos(\theta) K_y}{I_{xy}}$$

$$FirstOrderODEsP_4 := \sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\phi\right) = ((1-\beta)\,\Omega_\theta - \cos(\psi)\,\Omega_\phi\,(1+\beta))\,\Omega_\psi + \pi P \Bigg(\frac{1}{2} \frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2 - a^2)\,(1 - A_C)}{\sin(\alpha)} + \frac{1}{2}\cos(\alpha)\,(A_C + 3) \left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2 - a^2)\Bigg) (\sin(\phi)\,\pi_Y + \pi_X \cos(\phi))\,(\pi_X \sin(\phi)\sin(\psi) - \pi_Y \sin(\psi)\cos(\phi) + \pi_Z \cos(\psi)) \Big/ I_{xy} + P\,\pi\, a^2\, h\,(1 - A_T)\,(-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))\,(\sin(\phi)\,\pi_Y + \pi_X \cos(\phi)) \Big/ I_{xy}$$

```
select( has, op(2,FirstOrderODEsP[5]), K[x] );
location(FirstOrderODEsP[5],"):
FirstOrderODEsP[5] := subsop( "=rhs(tmp2)/I[xy]+rhs(tmp2T)/I[xy],
FirstOrderODEsP[5] );
```

$$\frac{-K_y \sin(\theta) + \cos(\theta) K_x}{I_{xy}}$$

$$FirstOrderODEsP_5 := \frac{\partial}{\partial t} \Omega_\psi = \left(\beta \cos(\psi)\,\Omega_\phi{}^2 + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \pi P\,(\pi_X \sin(\phi)\sin(\psi) - \pi_Y \sin(\psi)\cos(\phi) + \pi_Z \cos(\psi))\,(\pi_X \cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y \cos(\phi)) \Bigg(\frac{1}{2} \frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2 - a^2)\,(1 - A_C)}{\sin(\alpha)} + \frac{1}{2}\cos(\alpha)\,(A_C + 3) \left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2 - a^2)\Bigg) \Big/ I_{xy} + P\,\pi\, a^2\, h\,(1 - A_T)\,(-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))$$

$$(\pi_X \cos(\psi) \sin(\phi) - \sin(\psi) \pi_Z - \cos(\psi) \pi_Y \cos(\phi)) \Big/ I_{xy}$$

```
select( has, op(2,FirstOrderODEsP[6]), K[x] );
location(FirstOrderODEsP[6],"):
FirstOrderODEsP[6] :=
    subsop( "=-(rhs(tmp1)/I[xy]+rhs(tmp1T)/I[xy])*cos(psi),
FirstOrderODEsP[6] );
```

$$\frac{-\frac{\sin(\psi) K_z}{-1+\beta} - (K_x \sin(\theta) + \cos(\theta) K_y) \cos(\psi)}{I_{xy}}$$

$$FirstOrderODEsP_6 := \sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\theta\right) = ((\cos(\psi)^2 \beta + 1) \Omega_\phi + (-1+\beta) \cos(\psi) \Omega_\theta) \Omega_\psi$$
$$- \left(\pi P \left(\frac{1}{2} \frac{(\cos(\alpha)^2 - 2 \sin(\alpha)^2) (\sin(\alpha) h + \cos(\alpha) a) (b^2 - a^2) (1 - A_C)}{\sin(\alpha)}\right.\right.$$
$$\left. + \frac{1}{2} \cos(\alpha) (A_C + 3) \left(\frac{2}{3} \frac{b-a}{\sin(\alpha)} - \cos(\alpha) h - \frac{\cos(\alpha)^2 a}{\sin(\alpha)}\right) (b^2 - a^2)\right)$$
$$(\sin(\phi) \pi_Y + \pi_X \cos(\phi)) (\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) \Big/ I_{xy} + P \pi$$
$$a^2 h (1 - A_T) (-\pi_X \sin(\phi) \sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi) \cos(\phi))$$
$$\left.(\sin(\phi) \pi_Y + \pi_X \cos(\phi)) \Big/ I_{xy}\right) \cos(\psi)$$

The first-order ODEs can be cleaned up a bit.

```
foo := FirstOrderODEsP[4..6];
```

$$foo := \left[\sin(\psi) \left(\frac{\partial}{\partial t} \Omega_\phi\right) = ((1-\beta) \Omega_\theta - \cos(\psi) \Omega_\phi (1+\beta)) \Omega_\psi + \pi P \left(\right.\right.$$
$$\frac{1}{2} \frac{(\cos(\alpha)^2 - 2 \sin(\alpha)^2) (\sin(\alpha) h + \cos(\alpha) a) (b^2 - a^2) (1 - A_C)}{\sin(\alpha)}$$
$$\left. + \frac{1}{2} \cos(\alpha) (A_C + 3) \left(\frac{2}{3} \frac{b-a}{\sin(\alpha)} - \cos(\alpha) h - \frac{\cos(\alpha)^2 a}{\sin(\alpha)}\right) (b^2 - a^2)\right)$$
$$(\sin(\phi) \pi_Y + \pi_X \cos(\phi)) (\pi_X \sin(\phi) \sin(\psi) - \pi_Y \sin(\psi) \cos(\phi) + \pi_Z \cos(\psi)) \Big/ I_{xy} + P \pi$$
$$a^2 h (1 - A_T) (-\pi_X \sin(\phi) \sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi) \cos(\phi))$$

$$(\sin(\phi)\,\pi_Y + \pi_X\cos(\phi)) \Big/ I_{xy},\ \frac{\partial}{\partial t}\Omega_\psi = \left(\beta\cos(\psi)\,\Omega_\phi{}^2 + (-1+\beta)\,\Omega_\theta\,\Omega_\phi\right)\sin(\psi) + \pi P$$

$$(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))$$

$$(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y\cos(\phi))\Bigg($$

$$\frac{1}{2}\frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2 - a^2)\,(1 - A_C)}{\sin(\alpha)}$$

$$+\frac{1}{2}\cos(\alpha)\,(A_C + 3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2 - a^2)\Bigg) \Big/ I_{xy} + P\,\pi\,a^2\,h$$

$$(1 - A_T)\,(-\pi_X\sin(\phi)\sin(\psi) - \pi_Z\cos(\psi) + \pi_Y\sin(\psi)\cos(\phi))$$

$$(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y\cos(\phi)) \Big/ I_{xy},\ \sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) =$$

$$((\cos(\psi)^2\,\beta + 1)\,\Omega_\phi + (-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi - \Bigg(\pi P\Bigg($$

$$\frac{1}{2}\frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2 - a^2)\,(1 - A_C)}{\sin(\alpha)}$$

$$+\frac{1}{2}\cos(\alpha)\,(A_C + 3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2 - a^2)\Bigg)$$

$$(\sin(\phi)\,\pi_Y + \pi_X\cos(\phi))\,(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi)) \Big/ I_{xy} + P\,\pi$$

$$a^2\,h\,(1 - A_T)\,(-\pi_X\sin(\phi)\sin(\psi) - \pi_Z\cos(\psi) + \pi_Y\sin(\psi)\cos(\phi))$$

$$(\sin(\phi)\,\pi_Y + \pi_X\cos(\phi)) \Big/ I_{xy}\Bigg)\cos(\psi)\Bigg]$$

Select the main geometrical factor, which we will see is common to all three equations.

```
select_G := proc( expr )
  select( has, expr, b-a ):
  while has(",pi[X]) do
    select( has, ", b-a );
  od:
end:
```

```
G1 := select_G( rhs(FirstOrderODEsP[4]) ):
G2 := select_G( rhs(FirstOrderODEsP[5]) ):
G3 := select_G( rhs(FirstOrderODEsP[6]) ):
```

```
simplify(G1-G3);
simplify(G1-G2);
```

$$0$$

$$0$$

```
location( foo[1], G1 );
location( foo[2], G2 );
location( foo[3], G3 );
```

$$[2, 2, 3]$$

$$[2, 2, 5]$$

$$[2, 2, 2, 1, 3]$$

```
G[C](a,b,h,alpha,A[C]) = Pi*P/I[xy]*G1;
```

$$G_C(a, b, h, \alpha, A_C) = \pi P \left(\frac{1}{2} \frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)(\sin(\alpha) h + \cos(\alpha) a)(b^2 - a^2)(1 - A_C)}{\sin(\alpha)} + \frac{1}{2}\cos(\alpha)(A_C + 3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha) h - \frac{\cos(\alpha)^2 a}{\sin(\alpha)}\right)(b^2 - a^2)\right) \Big/ I_{xy}$$

```
G[T](a,h,A[T]) = Pi*P/I[xy]*a^2*h*(1-A[T]);
```

$$G_T(a, h, A_T) = \frac{\pi P a^2 h (1 - A_T)}{I_{xy}}$$

```
Gsubs := ["","]:
```

```
foo[1] := subs( G1=G[C](a,b,h,alpha,A[C])/Pi/P*I[xy],
                1-A[T]=G[T](a,h,A[T])/Pi/P*I[xy]/h/a^2,  foo[1] ):
foo[2] := subs( G2=G[C](a,b,h,alpha,A[C])/Pi/P*I[xy],
                1-A[T]=G[T](a,h,A[T])/Pi/P*I[xy]/h/a^2, foo[2] ):
foo[3] := subs( G3=G[C](a,b,h,alpha,A[C])/Pi/P*I[xy],
                1-A[T]=G[T](a,h,A[T])/Pi/P*I[xy]/h/a^2,  foo[3] ):
```

```
foo;
```

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right) = ((1-\beta)\Omega_\theta - \cos(\psi)\Omega_\phi(1+\beta))\Omega_\psi + G_C(a, b, h, \alpha, A_C)(\sin(\phi)\pi_Y + \pi_X\cos(\phi))(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi)) + G_T(a, h, A_T)(-\pi_X\sin(\phi)\sin(\psi) - \pi_Z\cos(\psi) + \pi_Y\sin(\psi)\cos(\phi))(\sin(\phi)\pi_Y + \pi_X\cos(\phi)), \frac{\partial}{\partial t}\Omega_\psi = \left(\beta\cos(\psi)\Omega_\phi^2 + (-1+\beta)\Omega_\theta\Omega_\phi\right)\sin(\psi) + (\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\pi_Z - \cos(\psi)\pi_Y\cos(\phi))G_C(a, b, h, \alpha, A_C) + G_T(a, h, A_T)\right.$$

$$(-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))$$

$$(\pi_X \cos(\psi)\sin(\phi) - \sin(\psi)\pi_Z - \cos(\psi)\pi_Y \cos(\phi)), \sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right) =$$

$$((\cos(\psi)^2 \beta + 1)\Omega_\phi + (-1+\beta)\cos(\psi)\Omega_\theta)\Omega_\psi - (G_C(a, b, h, \alpha, A_C)$$

$$(\sin(\phi)\pi_Y + \pi_X \cos(\phi))(\pi_X \sin(\phi)\sin(\psi) - \pi_Y \sin(\psi)\cos(\phi) + \pi_Z \cos(\psi)) +$$

$$G_T(a, h, A_T)(-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))$$

$$(\sin(\phi)\pi_Y + \pi_X \cos(\phi)))\cos(\psi)\Big]$$

```
select_K := proc( expr )
  select( has, expr, A[T] ):
  while has(",G[C]) do
    select( has, ", A[T] );
  od;
  ";
end:
```

```
g1 := select_K( rhs(foo[1]) )/G[T](a,h,A[T]);
g2 := select_K( rhs(foo[2]) )/G[T](a,h,A[T]);
g3 := select_K( rhs(foo[3]) )/G[T](a,h,A[T]);
```

$$g1 := (-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))(\sin(\phi)\pi_Y + \pi_X \cos(\phi))$$

$$g2 := (-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))$$

$$(\pi_X \cos(\psi)\sin(\phi) - \sin(\psi)\pi_Z - \cos(\psi)\pi_Y \cos(\phi))$$

$$g3 := (-\pi_X \sin(\phi)\sin(\psi) - \pi_Z \cos(\psi) + \pi_Y \sin(\psi)\cos(\phi))(\sin(\phi)\pi_Y + \pi_X \cos(\phi))$$

```
simplify(g1-g3);
```

$$0$$

```
gsubs := [ g[1](phi,psi)=g1, g[2](phi,psi)=g2, g[3](phi,psi)=g3 ]:
```

```
Ksubs := []:
for ii from 1 to 3 do
  remove( has, rhs(foo[ii]), beta );
  K.ii := factor("/g.ii*g[ii](phi,psi));
  print( K[ii] = " );
  Ksubs := [op(Ksubs), K[ii](a,b,h,alpha,A[C],A[T],phi,psi) = " ];

od:
```

$$K_1 = (-G_C(a, b, h, \alpha, A_C) + G_T(a, h, A_T))\, g_1(\phi, \psi)$$

$$K_2 = (-G_C(a, b, h, \alpha, A_C) + G_T(a, h, A_T))\, g_2(\phi, \psi)$$

$$K_3 = -(-G_C(a, b, h, \alpha, A_C) + G_T(a, h, A_T))\cos(\psi)\, g_3(\phi, \psi)$$

```
for ii from 1 to 3 do
```

```
  foo[ii] := lhs(foo[ii]) = select( has, rhs(foo[ii]), beta ) +
K[ii](a,b,h,alpha,A[C],A[T],phi,psi);
od:
```

Hence,

```
mat(foo);
```

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right)=((1-\beta)\,\Omega_{\theta}-\cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi}+K_1(a,b,h,\alpha,A_C,A_T,\phi,\psi)\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\psi}=\left(\beta\cos(\psi)\,\Omega_{\phi}{}^{2}+(-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi)+K_2(a,b,h,\alpha,A_C,A_T,\phi,\psi)\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right)=\right.$$

$$\left.((\cos(\psi)^2\,\beta+1)\,\Omega_{\phi}+(-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi}+K_3(a,b,h,\alpha,A_C,A_T,\phi,\psi)\right]$$

Check:

```
for ii from 1 to 3 do
  simplify( FirstOrderODEsP[ii+3] -
subs(op(Ksubs),op(gsubs),op(Gsubs),foo[ii]) );
od;
```

$$0=0$$

$$0=0$$

$$0=0$$

So the first order ODEs can be written

```
FirstOrderODEs := copy(FirstOrderODEsP):
for ii from 1 to 3 do
  FirstOrderODEs[ii+3] := foo[ii];
od:
mat(FirstOrderODEs);
```

$$\left[\frac{\partial}{\partial t}\phi=\Omega_{\phi}\right]$$

$$\left[\frac{\partial}{\partial t}\psi=\Omega_{\psi}\right]$$

$$\left[\frac{\partial}{\partial t}\phi=\Omega_{\phi}\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\phi}\right)=((1-\beta)\,\Omega_{\theta}-\cos(\psi)\,\Omega_{\phi}\,(1+\beta))\,\Omega_{\psi}+K_1(a,b,h,\alpha,A_C,A_T,\phi,\psi)\right]$$

$$\left[\frac{\partial}{\partial t}\Omega_{\psi}=\left(\beta\cos(\psi)\,\Omega_{\phi}{}^{2}+(-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi)+K_2(a,b,h,\alpha,A_C,A_T,\phi,\psi)\right]$$

$$\left[\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right)=\right.$$

$$((\cos(\psi)^2\,\beta+1)\,\Omega_\phi+(-1+\beta)\cos(\psi)\,\Omega_\theta)\,\Omega_\psi+K_3(a,b,h,\alpha,A_C,A_T,\phi,\psi)\Big]$$

where

```
mat(Ksubs);
mat(gsubs);
mat(Gsubs);
```

$$\begin{bmatrix} K_1(a,b,h,\alpha,A_C,A_T,\phi,\psi)=(-G_C(a,b,h,\alpha,A_C)+G_T(a,h,A_T))\,g_1(\phi,\psi) \\ K_2(a,b,h,\alpha,A_C,A_T,\phi,\psi)=(-G_C(a,b,h,\alpha,A_C)+G_T(a,h,A_T))\,g_2(\phi,\psi) \\ K_3(a,b,h,\alpha,A_C,A_T,\phi,\psi)=-(-G_C(a,b,h,\alpha,A_C)+G_T(a,h,A_T))\cos(\psi)\,g_3(\phi,\psi) \end{bmatrix}$$

$$[g_1(\phi,\psi)=(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))\,(\sin(\phi)\,\pi_Y+\pi_X\cos(\phi))$$

$$]$$

$$[g_2(\phi,\psi)=(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))$$

$$(\pi_X\cos(\psi)\sin(\phi)-\sin(\psi)\,\pi_Z-\cos(\psi)\,\pi_Y\cos(\phi))]$$

$$[g_3(\phi,\psi)=(-\pi_X\sin(\phi)\sin(\psi)-\pi_Z\cos(\psi)+\pi_Y\sin(\psi)\cos(\phi))\,(\sin(\phi)\,\pi_Y+\pi_X\cos(\phi))$$

$$]$$

$$\Bigg[G_C(a,b,h,\alpha,A_C)=\pi P\Bigg(\frac{1}{2}\frac{(\cos(\alpha)^2-2\sin(\alpha)^2)\,(\sin(\alpha)\,h+\cos(\alpha)\,a)\,(b^2-a^2)\,(1-A_C)}{\sin(\alpha)}+\frac{1}{2}\cos(\alpha)\,(A_C+3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)}-\cos(\alpha)\,h-\frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2-a^2)\Bigg)\Big/\,I_{xy}\Bigg]$$

$$\left[G_T(a,h,A_T)=\frac{\pi P\,a^2\,h\,(1-A_T)}{I_{xy}}\right]$$

```
latex(mat(FirstOrderODEs),`d:/dynamics/precession/FirstOrderODEs.tex`);
latex(mat(Ksubs),`d:/dynamics/precession/K_subs.tex`);
latex(mat(gsubs),`d:/dynamics/precession/gsubs.tex`);
latex(mat(Gsubs),`d:/dynamics/precession/G_subs.tex`);
```

Substitutions for Computer Code

```
FirstOrderODEs;
```

$$\Bigg[\frac{\partial}{\partial t}\phi=\Omega_\phi,\ \frac{\partial}{\partial t}\psi=\Omega_\psi,\ \frac{\partial}{\partial t}\phi=\Omega_\phi,$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_\phi\right)=((1-\beta)\,\Omega_\theta-\cos(\psi)\,\Omega_\phi\,(1+\beta))\,\Omega_\psi+K_1(a,b,h,\alpha,A_C,A_T,\phi,\psi),$$

$$\frac{\partial}{\partial t}\Omega_{\psi} = \left(\beta \cos(\psi)\,\Omega_{\phi}^{2} + (-1+\beta)\,\Omega_{\theta}\,\Omega_{\phi}\right)\sin(\psi) + K_2(a, b, h, \alpha, A_C, A_T, \phi, \psi),$$

$$\sin(\psi)\left(\frac{\partial}{\partial t}\Omega_{\theta}\right) =$$

$$((\cos(\psi)^2\,\beta + 1)\,\Omega_{\phi} + (-1+\beta)\cos(\psi)\,\Omega_{\theta})\,\Omega_{\psi} + K_3(a, b, h, \alpha, A_C, A_T, \phi, \psi)\Big]$$

```
subslist := [cos(alpha)=calpha, sin(alpha)=salpha, cos(phi)=cphi,
sin(phi)=sphi,
             cos(psi)=cpsi, sin(psi)=spsi, pi[X]=pX, pi[Y]=pY, pi[Z]=pZ,

             diff(Omega[phi],t)=phiddot, diff(Omega[psi],t)=psiddot,
             diff(Omega[theta],t)=thetaddot,
             Omega[phi]=phidot, Omega[psi]=psidot,
Omega[theta]=thetadot,
             A[C]=AC, A[T]=AT ]:
```

```
subs( subslist, FirstOrderODEs[4..6] );
```

$$[$$

$$spsi\ phiddot = ((1-\beta)\ thetadot - cpsi\ phidot\ (1+\beta))\ psidot + K_1(a, b, h, \alpha, AC, AT, \phi, \psi)$$

$$, psiddot = (\beta\ cpsi\ phidot^2 + (-1+\beta)\ thetadot\ phidot)\ spsi + K_2(a, b, h, \alpha, AC, AT, \phi, \psi),$$

$$spsi\ thetaddot =$$

$$((cpsi^2\,\beta + 1)\ phidot + (-1+\beta)\ cpsi\ thetadot)\ psidot + K_3(a, b, h, \alpha, AC, AT, \phi, \psi)]$$

```
subs( subslist, Ksubs );
```

$$[K_1(a, b, h, \alpha, AC, AT, \phi, \psi) = (-G_C(a, b, h, \alpha, AC) + G_T(a, h, AT))\ g_1(\phi, \psi),$$

$$K_2(a, b, h, \alpha, AC, AT, \phi, \psi) = (-G_C(a, b, h, \alpha, AC) + G_T(a, h, AT))\ g_2(\phi, \psi),$$

$$K_3(a, b, h, \alpha, AC, AT, \phi, \psi) = -(-G_C(a, b, h, \alpha, AC) + G_T(a, h, AT))\ cpsi\ g_3(\phi, \psi)]$$

```
subs( subslist, gsubs );
```

$$[g_1(\phi, \psi) = (-pX\ sphi\ spsi - pZ\ cpsi + pY\ spsi\ cphi)\ (sphi\ pY + pX\ cphi),$$

$$g_2(\phi, \psi) = (-pX\ sphi\ spsi - pZ\ cpsi + pY\ spsi\ cphi)\ (pX\ cpsi\ sphi - spsi\ pZ - cpsi\ pY\ cphi),$$

$$g_3(\phi, \psi) = (-pX\ sphi\ spsi - pZ\ cpsi + pY\ spsi\ cphi)\ (sphi\ pY + pX\ cphi)]$$

```
subs( subslist, Gsubs );
```

$$\Big[G_C(a, b, h, \alpha, AC) = \pi P\Big($$

$$\frac{1}{2}\,\frac{(calpha^2 - 2\ salpha^2)\ (salpha\ h + calpha\ a)\ (b^2 - a^2)\ (1 - AC)}{salpha}$$

$$+\frac{1}{2}\,calpha\,(AC+3)\left(\frac{2}{3}\frac{b-a}{salpha}-calpha\,h-\frac{calpha^2\,a}{salpha}\right)(b^2-a^2)\Bigg)\Big/\,I_{xy},$$

$$G_T(a,h,AT)=\frac{\pi\,P\,a^2\,h\,(1-AT)}{I_{xy}}\Bigg]$$

Fast Spin Approximation

Assume Ω_θ is large. Then

```
subslist := [ op(Ksubs), op(gsubs), diff(phi,t)=Omega[phi],
diff(psi,t)=Omega[psi],

-G[C](a,b,h,alpha,A[C])+G[T](a,h,A[T])=G(a,b,h,alpha,A[C],A[T])]:
```

```
explist := [Omega[phi],Omega[psi],G(a,b,h,alpha,A[C],A[T])]:
```

```
d2list := [ diff(Omega[theta],t,t)=D2theta, diff(Omega[phi],t,t)=D2phi,
            diff(Omega[psi],t,t)=D2psi ]:
```

```
subs( subslist,
      convert( diff( subs(subslist,subs(subslist,FirstOrderODEs[4..6])), t
), diff ) ):
```

```
convert( expansion( ", explist, 1 ), diff );
```

$$\Bigg[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\phi\right)=(1-\beta)\left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi+(1-\beta)\,\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right),$$

$$\frac{\partial^2}{\partial t^2}\Omega_\psi=(-1+\beta)\left(\frac{\partial}{\partial t}\Omega_\theta\right)\sin(\psi)\,\Omega_\phi+(-1+\beta)\,\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\phi\right)\sin(\psi),$$

$$\cos(\psi)\,\Omega_\psi\left(\frac{\partial}{\partial t}\Omega_\theta\right)+\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\theta\right)=$$

$$(-1+\beta)\cos(\psi)\left(\frac{\partial}{\partial t}\Omega_\theta\right)\Omega_\psi+(-1+\beta)\cos(\psi)\,\Omega_\theta\left(\frac{\partial}{\partial t}\Omega_\psi\right)\Bigg]$$

```
latex(mat(\"),`d:/dynamics/precession/FastEqs1.tex`);
```

```
foo := ":
```

Substitute the first-order ODEs into the rhs, expanding and simplifying, to get

```
subs( d2list, foo ):
subs( diff(Omega[theta],t)=solve(FirstOrderODEs[6],diff(Omega[theta],t)),
      diff(Omega[psi],t)=rhs(FirstOrderODEs[5]),
      diff(Omega[phi],t)=solve(FirstOrderODEs[4],diff(Omega[phi],t)),
      " ):
subs(subslist,"):
subs(subslist,"):
convert(expansion(",explist,1),diff):
invsubs( d2list, " ):
foox := collect( ", [Omega[psi],Omega[phi]], factor ):
```

```
mat(foox);
```

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\phi\right) = -(-1+\beta)^2\,\Omega_\theta{}^2\sin(\psi)\,\Omega_\phi + (-1+\beta)\,\Omega_\theta\,\mathrm{G}(a, b, h, \alpha, A_C, A_T)\right.$$
$$(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))$$
$$\left.(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y\cos(\phi))\right]$$

$$\left[\frac{\partial^2}{\partial t^2}\Omega_\psi = -(-1+\beta)^2\,\Omega_\theta{}^2\,\Omega_\psi - (-1+\beta)\,\Omega_\theta\,\mathrm{G}(a, b, h, \alpha, A_C, A_T)\right.$$
$$\left.(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))\,(\sin(\phi)\,\pi_Y + \pi_X\cos(\phi))\right]$$

$$\left[\sin(\psi)\left(\frac{\partial^2}{\partial t^2}\Omega_\theta\right) = (-1+\beta)^2\cos(\psi)\,\Omega_\theta{}^2\sin(\psi)\,\Omega_\phi - (-1+\beta)\cos(\psi)\,\Omega_\theta\right.$$
$$\mathrm{G}(a, b, h, \alpha, A_C, A_T)\,(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))$$
$$\left.(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y\cos(\phi))\right]$$

```
latex(",`d:/dynamics/precession/FastEqs2.tex`);
```

Since Ω_θ is large, we may assume it is slowly varying. Hence, in the third equation we set $\frac{\partial^2}{\partial t^2}\Omega_\theta = 0$ and therefore

```
isolate(rhs(foox[3]),Omega[phi]);
```

$$\Omega_\phi = \mathrm{G}(a, b, h, \alpha, A_C, A_T)\,(\pi_X\sin(\phi)\sin(\psi) - \pi_Y\sin(\psi)\cos(\phi) + \pi_Z\cos(\psi))$$
$$(\pi_X\cos(\psi)\sin(\phi) - \sin(\psi)\,\pi_Z - \cos(\psi)\,\pi_Y\cos(\phi)) \Big/ \left((-1+\beta)\,\Omega_\theta\sin(\psi)\right)$$

```
latex(",`d:/dynamics/precession/OmegaPhi1.tex`);
```

Assume π_X and π_Y are small. Then

```
expansion(",[pi[X],pi[Y]],1):
collect( ", [G(a,b,h,alpha,A[C],A[T]),pi[Z]], factor );
```

$$\Omega_\phi = \left(-\frac{\pi_Z{}^2\cos(\psi)}{(-1+\beta)\,\Omega_\theta} - \frac{(\sin(\psi)-\cos(\psi))\,(\sin(\psi)+\cos(\psi))\,(\sin(\phi)\,\pi_X - \cos(\phi)\,\pi_Y)\,\pi_Z}{(-1+\beta)\,\Omega_\theta\sin(\psi)}\right)$$
$$\mathrm{G}(a, b, h, \alpha, A_C, A_T)$$

```
latex(",`d:/dynamics/precession/OmegaPhi2.tex`);
```

Written out fully, we have

```
invsubs(subslist,"):
subs(Gsubs,subslist,");
```

$$\Omega_\phi = \left(-\frac{\pi_Z^{\,2}\cos(\psi)}{(-1+\beta)\,\Omega_\theta} - \frac{(\sin(\psi)-\cos(\psi))\,(\sin(\psi)+\cos(\psi))\,(\sin(\phi)\,\pi_X - \cos(\phi)\,\pi_Y)\,\pi_Z}{(-1+\beta)\,\Omega_\theta\,\sin(\psi)}\right)\Bigg($$
$$-\pi P\left(\frac{1}{2}\frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2-a^2)\,(1-A_C)}{\sin(\alpha)}\right.$$
$$\left.+\frac{1}{2}\cos(\alpha)\,(A_C+3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2-a^2)\right)\Big/ I_{xy}$$
$$\left.+\frac{\pi P a^2 h\,(1-A_T)}{I_{xy}}\right)$$

```
latex(",`d:/dynamics/precession/OmegaPhi3.tex`);
```

Precession Null

Numerical

We can determine the approximate angles α at which the precession is zero.

```
foo := select(has,rhs(''),P);
```

$$foo := -\pi P\left(\frac{1}{2}\frac{(\cos(\alpha)^2 - 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(b^2-a^2)\,(1-A_C)}{\sin(\alpha)}\right.$$
$$\left.+\frac{1}{2}\cos(\alpha)\,(A_C+3)\left(\frac{2}{3}\frac{b-a}{\sin(\alpha)} - \cos(\alpha)\,h - \frac{\cos(\alpha)^2\,a}{\sin(\alpha)}\right)(b^2-a^2)\right)\Big/ I_{xy}$$
$$+\frac{\pi P a^2 h\,(1-A_T)}{I_{xy}}$$

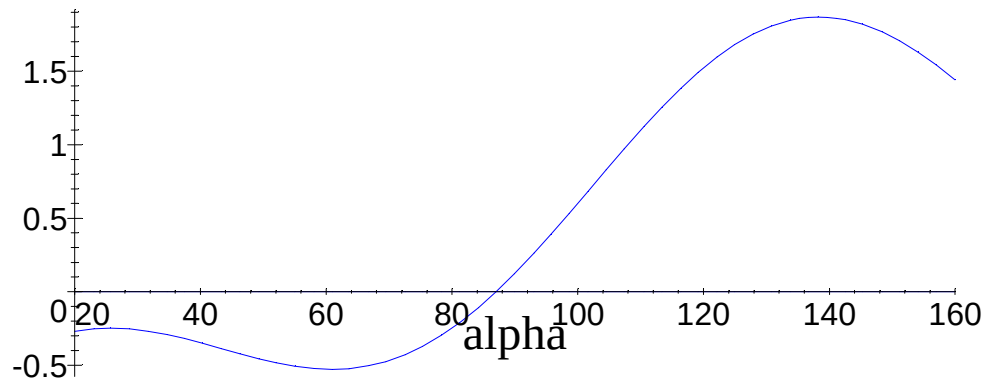
```
Collect( foo*I[xy]/Pi/P*sin(alpha)/(b^2-a^2), [1-A[C],1-A[T]], factor
);
```

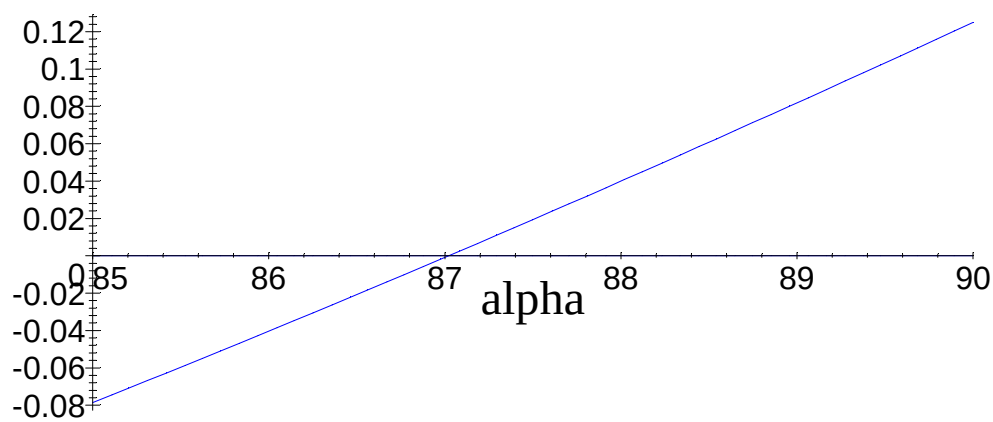
$$\frac{1}{2}(-\cos(\alpha)^2 + 2\sin(\alpha)^2)\,(\sin(\alpha)\,h + \cos(\alpha)\,a)\,(1-A_C) - \frac{\sin(\alpha)\,a^2 h\,(1-A_T)}{(a-b)\,(a+b)}$$
$$+\frac{1}{6}\cos(\alpha)\,(A_C+3)\,(2\,a - 2\,b + 3\cos(\alpha)\,h\sin(\alpha) + 3\cos(\alpha)^2\,a)$$

```
latex("=0,`d:/dynamics/precession/PrecessionNullEq.tex`);
```

```
Gfunc := fn(",alpha,a,b,h,A[C],A[T]):
```

```
plotsetup( plotdevice=jpeg,
plotoutput=`d:/dynamics/precession/PrecessionNull.jpg`,
           plotoptions=`height=768,width=1024` );
p1 := plot( {Gfunc(alpha*Pi/180,1,3,1,0.9,0.8),0}, alpha=20..160,
axes=normal ):
p1;
resetplot();
p1;
plot( {Gfunc(alpha*Pi/180,1,3,1,0.9,0.8),0}, alpha=85..90,
axes=normal );
```






```
fsolve( Gfunc(alpha*Pi/180,1,3,1,0.9,0.8), alpha, 85..95 );
```

$$87.02065404$$

Analytic

```
bar := Gfunc(alpha,a,b,h,A[C],A[T]);
```

$$bar := \frac{1}{2}\left(-\cos(\alpha)^2 + 2\sin(\alpha)^2\right)\left(\sin(\alpha)\,h + \cos(\alpha)\,a\right)\left(1 - A_C\right) - \frac{\sin(\alpha)\,a^2\,h\,(1 - A_T)}{(a-b)\,(a+b)}$$

$$+\frac{1}{6}\cos(\alpha)\,(A_C+3)\,(2\,a-2\,b+3\cos(\alpha)\,h\sin(\alpha)+3\cos(\alpha)^2\,a)$$

The exact solutions of this equation are arctangents of sixth-order polynomials — not a pretty sight. Let us instead try an iterative approach. We see that there is at minimum one useful solution, near $\alpha=\frac{\pi}{2}$. Try a solution of the form $\alpha=\frac{\pi}{2}+x_1\,\varepsilon+x_2\,\varepsilon^2$. Plugging into the equation, we find

```
collect( expansion( subs( alpha=Pi/2+x[1]*epsilon+x[2]*epsilon^2, bar
), epsilon, 2 ),
                    [epsilon,x[1],A[C]], factor );
```

$$\left(\left(\frac{5}{2}\,h\,A_C-\frac{1}{2}\,\frac{h\,(-b^2+a^2\,A_T)}{(a-b)\,(a+b)}\right)x_1^{\,2}+\frac{1}{3}\,x_2\,(2\,a+b)\,A_C-x_2\,(2\,a-b)\right)\varepsilon^2$$
$$+\left(\left(\frac{2}{3}\,a+\frac{1}{3}\,b\right)A_C-2\,a+b\right)x_1\,\varepsilon-h\,A_C+\frac{h\,(-b^2+a^2\,A_T)}{(a-b)\,(a+b)}$$

```
foo := ":
```

```
isolate( subs(epsilon^2=0,epsilon=1,"), x[1] ):
collect(",[h,A[C]],simplify);
subslist := ["]:
```

$$x_1=\frac{\left(A_C-\frac{-b^2+a^2\,A_T}{(a-b)\,(a+b)}\right)h}{\left(\frac{2}{3}\,a+\frac{1}{3}\,b\right)A_C-2\,a+b}$$

```
isolate( subs(subslist,coeff(foo,epsilon,2)), x[2] );
```

$$x_2=-\frac{\left(\frac{5}{2}\,h\,A_C-\frac{1}{2}\,\frac{h\,(-b^2+a^2\,A_T)}{(a-b)\,(a+b)}\right)\left(A_C-\frac{-b^2+a^2\,A_T}{(a-b)\,(a+b)}\right)^2h^2}{\left(\left(\frac{2}{3}\,a+\frac{1}{3}\,b\right)A_C-2\,a+b\right)^2\left(\frac{1}{3}\,(2\,a+b)\,A_C-2\,a+b\right)}$$

```
subslist := [op(subslist),"]:
```

```
latex(subslist[1],`d:/dynamics/precession/PrecessionNull_x1.tex`);
latex(subslist[2],`d:/dynamics/precession/PrecessionNull_x2.tex`);
```

Hence, we have, to second order, the result

```
alpha = Pi/2 + rhs(subslist[1]) + rhs(subslist[2]);
```

$$\alpha = \frac{1}{2}\pi + \frac{\left(A_C - \frac{-b^2 + a^2 A_T}{(a-b)(a+b)}\right)h}{\left(\frac{2}{3}a + \frac{1}{3}b\right)A_C - 2a + b} - \frac{\left(\frac{5}{2}h A_C - \frac{1}{2}\frac{h(-b^2 + a^2 A_T)}{(a-b)(a+b)}\right)\left(A_C - \frac{-b^2 + a^2 A_T}{(a-b)(a+b)}\right)^2 h^2}{\left(\left(\frac{2}{3}a + \frac{1}{3}b\right)A_C - 2a + b\right)^2 \left(\frac{1}{3}(2a+b)A_C - 2a + b\right)}$$

```
evalf(subs(a=1,b=3,h=1,A[C]=0.9,A[T]=0.8,")*180/Pi);
```

$$57.29577950\,\alpha = 87.03565960$$

```
evalf(subs(a=1,b=3,h=1,A[C]=0.9,A[T]=0.8,subslist)*180/Pi);
```

$$[57.29577950\,x_1 = -2.864788976,\ 57.29577950\,x_2 = -.09955141688]$$

```
restart;
alias( I=I ):
read `d:/dynamics/precession/precession.eqs`:
alias( psi=psi(t), theta=theta(t), phi=phi(t),
       Omega[x]=Omega[x](t), Omega[y]=Omega[y](t), Omega[z]=Omega[z](t),
       Omega[phi]=Omega[phi](t), Omega[psi]=Omega[psi](t),
       Omega[theta]=Omega[theta](t) ):
M := (a,e)->evalm(subs(alpha=a,eta=e,map(simplify,inverse(ConToCart)))):
```

```
save EulerEqs, r1, r2, r3, R, Obody, BodyEqs, SymTop,
     FirstOrderODEsK, FirstOrderODEsF, FFSymTop, KE,
     ConToCart, M, Pbody, cos_chi, xyz, r_cone, Fintegrand, Kintegrand,
     torque_coeffs, torque_cone, torque_integral_xyz, torque_xyz,
     PbodyT, cos_chiT, torque_xyzT, FirstOrderODEsP,
     Ksubs, Gsubs, gsubs, FirstOrderODEs,
     `d:/dynamics/precession/precession.eqs`;
```

Lagrangian Approach — Force-Free Motion