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MEAN AND APPARENT PLACE COMPUTATIONS IN THE NEW IAU SYSTEM. I. THE TRANSFORMATION OF ASTROMETRIC CATALOG SYSTEMS TO THE EQUINOX J2000.0

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ABSTRACT

Expressions are given for converting both observational and compiled catalogs from the old FK4 reference system of B1950.0 to the new FK5 reference system of J2000.0. A detailed discussion of the formal background is also given.

I. GENERAL REMARKS

In the course of our work, we have found it necessary to develop an approach to the transformation of existing observational catalogs of star positions, and compiled catalogs of positions and proper motions, to the equinox and epoch of J2000.0 in accordance with the conventions described in Resolution No. 1 of the Sixteenth General Assembly of the IAU at Grenoble (IAU 1977), referred to here as the IAU 1976 conventions. In surveying the literature, five approaches for accomplishing these reductions may be cited, namely those of Kaplan (1981), Standish (1983), Aoki *et al.* (1983), Lederle and Schwan (1984), and Yallop (1984). Software was developed to implement the approaches described by Kaplan and Standish. Neither approach produced results agreeing with the other, especially near the poles, until corrections to each had been introduced (Kaplan, Standish—private communications). Those corrections are described in the paper by Aoki *et al.* (1983).

There is a wide range of prospective users whose catalogs may be referred to arbitrary equinoxes and epochs, with associated zero points in right ascension and declination, and possibly with proper motions, not necessarily referred to the FK4 system. We present a discussion that permits flexibility of approach in order to tailor the software to a particular case and more easily to perform numerical experiments with alternative schemes. In our applications, this is preferable to an approach where the details of the operations necessary to perform a transformation from an arbitrary equator and equinox to that of J2000.0 are concealed in a single numerical matrix. A 6×6 matrix transformation is given in more detail in Paper II, which includes a comparison with this method. In general, we treat conventional values as exact and carry a minimum of 12 significant figures to avoid errors of precision near the celestial poles, and also to maintain at least nine significant figures in the final result.

The inertial character of star catalogs referred to the FK4 system, such as the AGK3 and SAO, may be improved by applying corrections for the following effects:

(1) Use the coefficients of the analytical functions given by the Astronomische Rechen-Institut (Schwan, to be published) to transform the positions and proper motions from the FK4 system to the "FK5" system. The "FK5" is placed in quotation marks to emphasize that it is the version resulting from a revision of the FK4 by the incorporation of new observational materials from transit circles and astrolabes, but still referred to the IAU 1968 system for constants and conventions. The "FK5" positions and proper motions are referred to the equator and equinox of B1950.0.

(2) Remove the elliptic terms in aberration from the catalog positions.

(3) Apply zero-point corrections to the right ascensions and proper motions in right ascension to correct for the equinox error of the FK4 and its motion (Fricke 1982).

(4) Correct the proper-motion system for the new precessional parameters (Fricke 1967, 1971; Lieske *et al.* 1977).

(5) Change the unit of time from tropical to Julian centuries.

The work of Aoki *et al.* and that of Standish culminate in the evaluation of a 6×6 matrix by which a six-component position and proper-motion vector may be multiplied to complete the transformation to the J2000.0 equinox and epoch. Kaplan, Lederle, and Schwan, on the other hand, have presented a combined vectorial and spherical trigonometric approach where corrections are applied to the spherical coordinates, and only the precession is accomplished by a matrix multiplication. Aoki *et al.* and Lederle and Schwan have taken the additional step of introducing the discontinuity in the right ascension system into the transformation procedure itself. This step was taken to ensure that the magnitude of the right ascension discontinuity at the beginning of 1984 would equal the value assumed in the discussion of the amended expression for Greenwich Mean Sidereal Time at 0^h UT, thus avoiding a discontinuity in UT (Aoki *et al.* 1982). In other words, the sole purpose for introducing the discontinuity in right ascension into the transformation procedure itself at the beginning of 1984 is to assure consistency with the basic premise of Aoki *et al.* (1982) that the right ascension of the FK4 equinox in the FK5 system at the beginning of 1984 is $0\overset{s}{.}06930$. Because of the special role played by the FK4 in the establishment of UT1, special consideration applies to this epoch. However, in the case of the transformation of observational catalogs from an arbitrary equinox to J2000.0, 1984 should play no special role.

Since many of the transformation parameters are themselves time dependent, selection of the most appropriate epoch for evaluation of these parameters must be carefully considered. That is, it is possible to introduce epoch- and position-dependent distortions into the transformed reference frame if an inappropriate epoch is chosen. However, this is likely to be a problem only for a few years, while conversion of old reference data to the new IAU 1976 basis is taking place. Ultimately, new observations will be available which will have been based on the IAU 1976 conventions at the time of observation and the problem will no longer affect the positions. Proper-motion systems, on the other hand, are likely to be adversely affected for some time into the future.

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II. DISCUSSION OF VARIOUS ASPECTS OF THE IAU 1976 CONVENTIONS

a) The Unit of Time

The primary unit of time has been the tropical year at the fundamental epoch of 1900 January $0^d\,12^h$ ET. The tropical year is defined as the interval during which the Sun's mean longitude, referred to the mean equinox of date, increases by 360°. From the Explanatory Supplement (ES), p. 69ff: The adopted measure of this unit is determined by the coefficient of T, measured in centuries of 36 525 Julian days, in Newcomb's expression for the geometric mean longitude of the Sun, referred to the mean equinox of date, namely

$$L = 279°41'48''.04 + 129602768''.13T + 1''.089T^2. \quad (1)$$

The tropical year at 1900 January $0^d\,12^h$ ET will accordingly contain

$$(360\times60\times60\times36525\times86400)/129602768.13$$
$$= 31556925.9747 \text{ ephemeris seconds.}$$

From this, the length of the tropical century may be deduced as 36524.21987817305 Julian days at 1900. The Julian date of the beginning of any Besselian year may be calculated from

$$\text{JD} = 2433282.42345905 + 365.2421987817305(\text{BY} - \text{B}1950.0), \quad (2)$$

where BY is the Besselian year.

The presence of the T^2 term in Eq. (1) implies that the coefficient of T in the same equation increases by 1″089 per century, and that the length of the tropical century is epoch dependent and decreases by $26^s.52$ after one Julian century. It is also clear that this equation for the length of the tropical century depends upon Newcomb's theory of the Sun. In critical cases, over long periods of time, it may be necessary to use a time-dependent coefficient of the (BY − B1950.0) term to determine the length of the tropical century in days and fractions of a day.

Proper motions, precessional motions, and other angular motions expressed in units of tropical centuries can be transformed to units of Julian centuries by multiplication by the factor

$$F = 36525/36524.2198782 = 1.00002135903. \quad (3)$$

b) FK4 Zero-Point Correction in Right Ascension

An early discussion of the problem of the position and the motion of Newcomb's equinox and the FK4 equinox can be found in Blackwell (1977). Later, it was determined by Fricke (1982) that the FK4 right ascension system requires a correction of $+0^s.035$ at B1950.0. This has been referred to as an equinox correction. It implies that the right ascension of every star in any catalog whose right ascensions are referred to the system of FK4 at epoch and equinox B1950.0 must be increased by $+0^s.035$. The equinox correction at any epoch T, given as a function of its value at B1950.0 and the difference in epoch $(T - T_0)$, has been given by Fricke (1982) as follows:

$$E(T) = 0^s.035 + 0^s.085(T - T_0)/36524.2198782, \quad (4)$$

where T is the Julian date at any epoch and T_0 is the Julian date corresponding to the beginning of the Besselian year B1950.0, i.e., $T_0 = 2433282.42345905$. We suppose that the divisor of the epoch difference should be the number of Julian days per tropical century, but no significant error is made if the number of Julian days per Julian century (36 525) is adopted instead.

Aoki *et al.* (1983) have made a convincing argument that, for the sake of consistency with the new definition of UT introduced at 0^h UT on 1984 January 1 (Aoki *et al.* 1982), it is important that the FK4 equinox correction should be applied to right ascensions of the FK4 catalog referred to the same equinox and epoch (1984 January 1) as the date on which the new definition went into effect. The equinox correction corresponding to that epoch is accordingly

$$E(2445700.5) = +0^s.06390.$$

The reduction to the IAU 1976 system of constants at J2000.0 of the right ascension system of an observational catalog (as distinct from a compiled catalog) referred to the FK4 system at an arbitrary equinox and at the mean epoch of observation should be done in a manner consistent with the way in which the catalog was referred to the system of FK4 at the time of observation. On the other hand, if an observational catalog has been constructed from a fundamental treatment of the observations of solar system objects and thus referred to the observed dynamical equinox, the reduction to the IAU 1976 system of constants at J2000.0 would ignore the zero-point correction normally required by the FK4 system in right ascension.

c) The Correction to the FK4 Proper-Motion System in Right Ascension

The existence of a secular term of the form $0^s.085(T - T_0)/36524.2198782$ in the equinox correction has generally been interpreted as a constant, nonprecession-dependent correction applicable to the FK4 right ascension proper-motion system (Fricke 1982). This represents the best understanding of the term available at the present time and implies that at any epoch, and regardless of the equinox, all right ascension proper motions of the FK4 catalog (or any catalog with the proper motions referred to the FK4 system) must be increased by $+0^s.085$ per tropical century.

The nonlinear character of precession viewed as a mathematical operation suggests that subtle differences can be introduced into a catalog transformation from one epoch and equinox to another, depending on the epoch at which corrections are introduced. A number of possibilities come to mind. One could, for example, apply the corrections to positions and proper motions referred to

(1) the epoch and equinox of the beginning of 1984; or

(2) the epoch and equinox of B1950.0; or

(3) the epoch and equinox of the mean epoch of observation.

Among these three options, the third one most closely approximates the operation as it would have been carried out if the improved IAU 1976 system had been available at the time of observation. In the case of individual observational catalogs such as the series of fundamentally observed Washington Six-inch Transit Circle catalogs, the reduction to the FK5 equinox is best done by directly calculating the difference FK5 − Catalog from the stars in common to both catalogs.

The problem of the reduction of a fundamentally observed transit-circle catalog to J2000.0 will be discussed in Sec. III on the transformation of observational catalogs.

d) Elliptic Terms in Aberration

1) Effect on position

Recommendation 4 of IAU 1976 Resolution No. 1 indicates that from 1984 onwards, stellar aberration is to be computed from the total velocity of the Earth referred to the barycenter of the solar system, and mean catalog places are not to contain elliptic terms of aberration.

The problem of the reduction of observational catalogs to a uniform system is complicated by the change in the conventional value of the constant of aberration, which since 1911 had been taken as 20″.47 (Paris conference 1911) and from 1968 has been taken as 20″.496 (IAU 1966). Before 1911, the value of Struve and Peters, 20″.4551, was in common use, and from 1984 the value of 20″.49552 has been introduced as the conventional value. Reduction to a uniform system is further complicated by changes in the method of computing the aberrational day numbers, C and D, given in the national ephemerides and almanacs. Before 1960, the aberrational day numbers C and D were calculated from a circular approximation to the motion of the Earth in which small periodic terms due to the action of the Moon and planets and the elliptic terms in aberration were neglected (see ES, p. 48ff.). Beginning with 1960, the aberrational day numbers C and D were derived from the true velocity of the Earth referred to the center of mass of the solar system and to a dynamically determined frame of reference. The elliptic terms in aberration due to the eccentricity of the Earth's orbit were then removed from an otherwise completely correct expression of the annual aberration. Furthermore, as of 1960, ephemeris values of the aberrational day numbers properly reflect the slowly changing eccentricity and longitude of perihelion of the Earth's orbit (ES, p. 158ff.).

When working with individual observational catalogs, reductions to a uniform system of annual aberration should take into account the changes in the value of the constant of aberration and the method of calculation of the published values of the aberrational day numbers, C and D. This will adjust the published values of the observed positions in a manner consistent with conventions in use at the time the observations were made and reduced from apparent to mean place. If this procedure is not followed, then corrupted data will be produced, and the potential for systematic improvement will be diminished.

The FK4 catalog, and other fundamental catalogs such as the N30 (Morgan 1952) and the General Catalog (Boss 1937), were compiled from observations made prior to 1960, when a circular approximation was made to the motion of the Earth in its orbit, and the elliptic terms in aberration were ignored, except for stars within 10° of the poles in the case of the FK4 catalog.

Due to the rapid, daily change in the value of the circular component of annual aberration, it is not possible to correct mean positions for changes that have occurred in the adopted value of the constant of aberration either in the compiled fundamental catalogs or in observational catalogs of mean positions. Observations of solar system objects are normally given on a daily basis and may be corrected with a high degree of rigor. However, because the elliptic terms in aberration change so very slowly, their influence can be removed from the FK4 catalog positions by subtracting the elliptic terms in aberration from the catalog right ascension and declination ($\alpha_{\text{cat}}, \delta_{\text{cat}}$) to obtain the corrected right ascension and declination (α,δ) as follows:

$$\alpha = \alpha_{\text{cat}} - (\Delta C \cos \alpha_{\text{cat}} + \Delta D \sin \alpha_{\text{cat}})/(15 \cos \delta_{\text{cat}}), \tag{5}$$

$$\delta = \delta_{\text{cat}} - (\Delta D \cos \alpha_{\text{cat}} - \Delta C \sin \alpha_{\text{cat}}) \sin \delta_{\text{cat}} - \Delta C \tan \epsilon \cos \delta_{\text{cat}}, \tag{6}$$

where $\Delta C = -0''.065838$, $\Delta D = +0''.335299$, and $\Delta C \tan \epsilon = -0''.028553$ at epoch B1950.0, using the J2000.0 value of the constant of aberration.

These equations represent the classical elliptic aberration with first-order accuracy in the ratio of the Earth's velocity to the velocity of light (Woolard and Clemence 1966, p. 113ff.). More than the necessary number of significant figures is carried in order to ensure correct rounding. For applications requiring accuracies better than 1 milliarcsecond (mas), the discussions of Stumpf (1979, 1980) regarding the second-order relativistic terms in elliptic aberration should be consulted (Paper IV).

The numerical values for ΔC, ΔD, and $\Delta C \tan \epsilon$ given here are not generally applicable to every case in which positions have been referred to the equinox of B1950.0. In the case of an observational catalog referred to an arbitrary equinox, each position (where right ascension and declination must be treated separately if their mean epochs of observation are not the same) may be precessed to the equinox of the mean epoch of observation and then corrected for the elliptic terms of aberration using values referred to the moving equinox (see, for example, Scott 1964). Many observational catalogs will require an epoch-dependent correction of this type. Compiled catalogs may or may not require a correction depending on how the compilation was done (Lederle and Schwan 1984).

The time-dependent expressions for the coefficients of Eqs. (5) and (6) may be written as follows (Woolard and Clemence 1966, p. 114ff):

$$\Delta C = -ke \cos \Gamma \cos \epsilon,$$
$$\Delta D = -ke \sin \Gamma, \tag{7}$$

where

$k = 20''.49552$, the constant of aberration at J2000.0,

e = eccentricity of the Earth's orbit

$$= 0.01673011 - 0.00004193(T - T_0) - 0.000000126(T - T_0)^2,$$

Γ = mean longitude of perigee of the solar orbit

$$= 282°04'49''.951 + 6190''.67(T - T_0) + 1''.65(T - T_0)^2 + 0''.012(T - T_0)^3,$$

ϵ = obliquity of the ecliptic

$$= 23°26'44''.836 - 46''.8495(T - T_0) - 0''.00319(T - T_0)^2 + 0''.00181(T - T_0)^3,$$

where $(T - T_0)$ is in units of Julian centuries of 36 525 days and T_0 is the epoch B1950.0 (2433282.42345905). Equations (7) have been developed from ES, p. 98, and may be used to refer Eqs. (5) and (6) to any epoch of observation relative to the moving (i.e., of date) equinox.

For recently observed Washington catalogs, such as the W5(50) Six-inch Transit Circle Catalog (1963–1971) (Hughes and Scott 1982) and the WL(50) Seven-inch Transit Circle Catalog (1967–1973) (Hughes, Smith, and Branham 1989) observed from El Leoncito in San Juan, Argentina, the assumed constant of aberration used in the apparent place computations was 20.″496. The W6(50) Six-inch Transit Circle Catalog (1973–1982) (currently under discussion) will be rereduced in strict accordance with the IAU 1976 resolution and will be renamed to reflect J2000.0 as the equinox to which it will be referred. A preliminary fundamental reduction of this catalog will be undertaken pending the publication of the FK5, at which time a definitive discussion will be possible wherein the zero point of the instrumental clock-star right ascension system will be referred as rigorously as possible to the FK5 system. It is anticipated that future observational programs at the U.S. Naval Observatory will involve determination of time of transit referred to UT and an atomic timescale.

2) Effect on proper motion

Aoki *et al.* (1983) introduce a correction to the proper motions to compensate for a secular change of the elliptic aberration associated with the variation of the eccentricity of the Earth's orbit, longitude of solar perigee, and obliquity of the ecliptic with time. Lederle and Schwan remark that this is unnecessary because of the practice followed in the compilation of FK4 proper motions by which differences in right ascension and declination between observed positions and positions computed on the basis of the FK3 were used. In fact, the problem is complicated by the practice of having eliminated the influence of elliptic aberration for circumpolar stars before precessing to a new equinox and epoch, and then reintroducing the effect of elliptic aberration appropriate for the new equinox and epoch. This was not done for the stars between $+80°$ and $-80°$ declination (FK3, 1937; FK4, 1963), which means that no single practice will work equally well or consistently for all stars of the FK4, nor for any catalogs referred to the FK4 system. Corrections for the secular change of elliptic terms in aberration are not necessary for proper motions of circumpolar FK4 stars, but a correction is required for FK4 stars between $+80°$ and $-80°$.

A very good case can be made that the highest-accuracy transformation of the FK4 catalog to the equinox of J2000.0 would result if the procedure outlined above were followed, paying special attention to the treatment of the stars as regards the elliptic terms in aberration. Stars within 10° of the poles need only have the elliptic terms in aberration at B1950.0 removed from their positions. Their proper motions need no modification. For stars between $+80°$ and $-80°$ in declination, the positions and proper motions should be precessed to the equinox of the mean epoch of observation and the elliptic terms in aberration (referred to the moving equinox of the mean epoch of observation given by Eq. (7)) should be removed from the positions. Equations (5) and (6) may be differentiated with respect to time to give the equations for the corrected proper motions (μ,μ') in right ascension and declination in terms of the catalog proper motions in right ascension and declination (μ_{cat},μ'_{cat}), and the catalog positions ($\alpha_{cat},\delta_{cat}$),

$$\begin{aligned}\mu = \mu_{cat} &- (\Delta\dot{C}\cos\alpha_{cat} + \Delta\dot{D}\sin\alpha_{cat})/(15\cos\delta_{cat}) - \mu_{cat}\sin 1''(-\Delta C\sin\alpha_{cat} + \Delta D\cos\alpha_{cat})/\cos\delta_{cat} \\ &- \mu'_{cat}\sin 1''(\Delta C\cos\alpha_{cat} + \Delta D\sin\alpha_{cat})\tan\delta_{cat}/(15\cos\delta_{cat}),\end{aligned} \tag{8}$$

$$\begin{aligned}\mu' = \mu'_{cat} &- (\Delta\dot{D}\cos\alpha_{cat} - \Delta\dot{C}\sin\alpha_{cat})\sin\delta_{cat} - 15\mu_{cat}\sin 1''(-\Delta D\sin\alpha_{cat} - \Delta C\cos\alpha_{cat})\sin\delta_{cat} \\ &- \mu'_{cat}\sin 1''(\Delta D\cos\alpha_{cat} - \Delta C\sin\alpha_{cat})\cos\delta_{cat} \\ &- (\Delta\dot{C}\tan\epsilon + \Delta C\dot{\epsilon}_r\sec^2\epsilon)\cos\delta_{cat} + 15\mu_{cat}\sin 1''\Delta C\tan\epsilon\sin\delta_{cat},\end{aligned} \tag{9}$$

where the factor 15 converts arc to time or vice versa

$$\begin{aligned}\Delta\dot{C} &= +ke\dot{\Gamma}_r\cos\epsilon\sin\Gamma - k(\dot{e}\cos\epsilon - e\dot{\epsilon}_r\sin\epsilon)\cos\Gamma, \\ \Delta\dot{D} &= -k\dot{e}\sin\Gamma - ke\dot{\Gamma}_r\cos\Gamma, \\ \Delta\dot{C}\tan\epsilon &= +ke\dot{\Gamma}_r\sin\epsilon\sin\Gamma - k(\dot{e}\cos\epsilon - e\dot{\epsilon}_r\sin\epsilon)\cos\Gamma\tan\epsilon,\end{aligned} \tag{10}$$

where $\dot{\Gamma}_r$ and $\dot{\epsilon}_r$ are the rates of change of Γ and ϵ in radians per tropical century.

The quantities $\Delta\dot{C}$, $\Delta\dot{D}$, and $\Delta\dot{C}\tan\epsilon$ have to be evaluated in a fixed frame at B1950.0. To convert from the rotating frame of date to this fixed frame, the instantaneous rates of change m_o, n_o, and p_o at B1950.0 have to be removed from the displacements in right ascension ($\dot{\Gamma}\cos\epsilon$), declination ($\dot{\Gamma}\sin\epsilon$), and longitude ($\dot{\Gamma}$) due to precession of the mean longitude of perigee. Hence, with

$$\dot{e} = \frac{d}{dT}e|_{T=0} = -0.00004193 \text{ per tropical century}, \quad m_o = 4609.''90 \text{ per tropical century},$$

$$\dot{\Gamma} = \frac{d}{dT}\Gamma|_{T=0} = +6190.''54 \text{ per tropical century}, \quad n_o = 2004.''26 \text{ per tropical century},$$

$$\dot{\epsilon} = \frac{d}{dT}\epsilon|_{T=0} = -46.''8485 \text{ per tropical century}, \quad p_o = 5026.''75 \text{ per tropical century},$$

the numerical values for $\Delta\dot{C}$, $\Delta\dot{D}$, and $\Delta\dot{C}\tan\epsilon$ in the fixed frame at B1950.0 are

$$\begin{aligned}\Delta\dot{C} &= -0.''001580 \text{ per tropical century}, \\ \Delta\dot{D} &= -0.''001245 \text{ per tropical century}, \\ \Delta\dot{C}\tan\epsilon &= -0.''000677 \text{ per tropical century}.\end{aligned}$$

The equations for m_o and n_o are given in Eq. (23) in Sec. II*f*, and $p_o = m_o\cos\epsilon + n_o\sin\epsilon$.

e) The Precession

1) Newcomb's precession

Since Newcomb's value of the precession constant (Newcomb 1898) came into general use, several representations have been used for the practical realization of the accumulated precession angles ζ_o, z, and θ. One may cite the formulations given in the Explanatory Supplement (1961), the development by Andoyer (1911) from which the discussion by Woolard and Clemence (1966) was drawn, and the formulation given in the introduction to the SAOC (1966) catalog, as well as a more recent discussion by Kinoshita (1975) repeated by Aoki *et al.* (1983), where the discussion is carried to one more significant figure than that given by Andoyer.

Among the various representations, differences in the accumulated precession angles of the order of 1 mas are found in as short a time as 30 yr. For example, a comparison of results from Woolard and Clemence (1966), Kinoshita (1975), and the ES for the accumulated precession angles ζ_o, z, and θ between equinox and equator of B1950.0 and the equinox and equator of 1984 January $1^d 0^h$ gives

	Woolard and Clemence	Kinoshita	ES	Adopted
ζ_o	783$''$.70925	783$''$.70938	783$''$.70798	783$''$.7092
z	783$''$.80093	783$''$.80106	783$''$.79942	783$''$.8009
θ	681$''$.38830	681$''$.38849	681$''$.38732	681$''$.3883

Results from the Kinoshita formulation agree with that of Woolard and Clemence at the 0.1 mas level. Disagreement of the results from the ES at the 1 to 2 mas level is brought about by the neglect of higher-order terms in the ES (p. 30).

The Andoyer (Woolard and Clemence) equations for the precession angles are

$$\zeta_o = (23035''.545 + 139''.720t_1 + 0''.060t_1^2)\tau + (30''.240 - 0''.270t_1)\tau^2 + 17''.995\tau^3,$$
$$z = (23035''.545 + 139''.720t_1 + 0''.060t_1^2)\tau + (109''.480 + 0''.390t_1)\tau^2 + 18''.325\tau^3, \quad (11)$$
$$\theta = (20051''.12 - 85''.29t_1 - 0''.37t_1^2)\tau + (-42''.65 - 0''.37t_1)\tau^2 - 41''.80\tau^3,$$

where t_1 is the interval in units of 1000 tropical years of 365242.198782 days measured by the differences between the initial epoch and B1850.0:

$$t_1 = [\text{epoch(initial)} - \text{B1850.0}]/(1000 \text{ tropical years}), \quad (12)$$

$$\tau = t_2 - t_1, \quad (13)$$

where t_2 is the interval in units of 1000 tropical years measured by the difference between the final epoch and B1850.0:

$$t_2 = [\text{epoch(final)} - \text{B1850.0}]/(1000 \text{ tropical years}). \quad (14)$$

Evaluating these expressions with $t_1 = 0.1$, $t_2 = 0.133999566814$, and $\tau = 0.033999566814$ gives the adopted values of the precession angles, which represents the precessional motion of the reference frame from the equinox and equator of B1950.0 to 1984 January $1^d 0^h$, and are given in the last column of the table above. They have been rounded to 0.1 mas. No special significance should be attached to the fact that the unit of time in the expressions used by Wollard and Clemence (1966) and given above is 1000 tropical years.

2) The IAU 1976 precession

The new basis for precession is taken directly from the discussion of Lieske *et al.* (1977) and modified by the small amount discussed in Lieske (1979):

$$\zeta_A = (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (0''.30188 - 0''.000344T)t^2 + 0''.017998t^3,$$
$$z_A = (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (1''.09468 + 0''.000066T)t^2 + 0''.018203t^3, \quad (15)$$
$$\theta_A = (2004''.3109 - 0''.85330T - 0''.000217T^2)t + (-0''.42665 - 0''.000217T)t^2 - 0''.041833t^3,$$

where T is the interval in units of Julian centuries of 36 525 days of Barycentric Dynamical Time (TDB) between the initial epoch and J2000.0,

$$T = [\text{epoch(initial)} - \text{J2000.0}]/36\,525, \quad (16)$$

and t is the interval in units of Julian centuries of 36 525 days between the final epoch and the initial epoch,

$$t = [\text{epoch(final)} - \text{epoch(initial)}]/36\,525. \quad (17)$$

Evaluating these expressions with $T = -0.1600136893$ and $t = -T$, the accumulated angles from 1984 January $1^d 0^h$ to J2000.0 are $\zeta_A = 368''.9985$, $z_A = 369''.0188$, and $\theta_A = 320''.7279$.

f) The Proper-Motion System

The discontinuity in the right ascension proper-motion system of the FK4 catalog is to be imposed at 1984 January $1^d 0^h$. Proper motions referred to that equinox and epoch should

(1) be corrected for the zero point of the right ascension proper-motion system as discussed in Sec. II*b*,

(2) be changed from the Newcomb to the IAU 1976 (Lieske *et al.* 1977) precession basis, and

(3) be expressed in units of Julian rather than tropical centuries.

Only points (2) and (3) apply to the proper motions in declination.

The fundamental condition to be satisfied at 1984 January $1^d 0^h$ is that the centennial variation on the new basis shall equal the centennial variation on the old basis plus a constant correction in the case of the right ascension proper motions:

$$\dot{\alpha}_{\text{new}} = \dot{\alpha}_{\text{old}} + \dot{E}, \quad \dot{\delta}_{\text{new}} = \dot{\delta}_{\text{old}}. \tag{18}$$

Let quantities without subscripts be taken as the new values referred to the IAU 1976 basis, and those with subscripts "o" be taken as referring to the old (Newcomb) basis, then in right ascension,

$$\mu + m + n \sin\alpha \tan\delta = (\mu_o + m_o + n_o \sin\alpha_o \tan\delta_o + \dot{E})F, \tag{19}$$

and in declination

$$\mu' + n \cos\alpha = (\mu'_o + n_o \cos\alpha_o)F, \tag{20}$$

where

- μ,μ' are the proper motions in right ascension and declination, respectively, referred to the IAU 1976 basis and precessed to 1984 January $1^d 0^h$.
- m,n are the centennial general precession in right ascension and declination, respectively, referred to the IAU 1976 basis at 1984 January $1^d 0^h$.
- α_o,δ_o are the right ascension and declination for the FK4 catalog precessed to 1984 January 1.
  - $\alpha = \alpha_o + 0\overset{s}{.}06390$.
  - $\delta = \delta_o$.
  - $\dot{E} = 0\overset{s}{.}085$ per tropical century.
- μ_o,μ'_o are the proper motions in right ascension and declination, respectively, from the FK4 catalog, for example, and precessed to 1984 January $1^d 0^h$.
- m_o,n_o are the centennial general precession in right ascension and declination, respectively, referred to Newcomb's precession at 1984 January $1^d\ 0^h$.
- F is the factor for converting from tropical centuries to Julian centuries.

The quantities m and n may be derived from the equations for the IAU 1976 precession in Sec. II*e2* as follows:

$$m = \frac{d}{dt}(\zeta_A + z_A)\,|_{t=0} = 4612\overset{''}{.}4362 + 2\overset{''}{.}79312T - 0\overset{''}{.}000278T^2,$$

$$n = \frac{d}{dt}(\theta_A)\,|_{t=0} = 2004\overset{''}{.}3109 - 0\overset{''}{.}85330T - 0\overset{''}{.}000217T^2. \tag{21}$$

At 1984 January $1^d 0^h$, $T = -0.1600136893$ Julian centuries and

$$m = 4611\overset{''}{.}98926 \text{ per Julian century} = 307\overset{s}{.}465950 \text{ per Julian century}, \tag{22a}$$

and

$$n = 2004\overset{''}{.}44743 \text{ per Julian century} = 133\overset{s}{.}629829 \text{ per Julian century}. \tag{22b}$$

Similarly, m_o and n_o may be derived from the equations for Newcomb's precession in Sec. II*e1* as follows:

$$m_o = \frac{d}{d\tau}(\zeta_o + z)\,|_{\tau=0} = 46071\overset{''}{.}090 + 279\overset{''}{.}440t_1 + 0\overset{''}{.}120t_1^2, \tag{23}$$

$$n_o = \frac{d}{d\tau}(\theta)\,|_{\tau=0} = 20051\overset{''}{.}12 - 85\overset{''}{.}29t_1 - 0\overset{''}{.}37t_1^2.$$

At 1984 January $1^d 0^h$, $t_1 = 0.13399956681$, in units of 1000 tropical centuries, and m_o and n_o expressed in Julian centuries, are

$$m_o = 4610\overset{''}{.}95218 \text{ per Julian century} = 307\overset{s}{.}396812 \text{ per Julian century}, \tag{24a}$$

and

$$n_o = 2004\overset{''}{.}01126 \text{ per Julian century} = 133\overset{s}{.}600750 \text{ per Julian century}. \tag{24b}$$

The values of m, n, m_o, and n_o will be used later in Sec. II*g6*.

g) Equations for the Transformation of Catalogs from B1950.0 to J2000.0

This transformation is a mean place to mean place conversion. That is, it transforms catalog mean places and proper motions from one reference epoch (B1950.0) to another (J2000.0). The complexity of the transformation results from the changes in constants, timescales, and procedures mandated by the IAU for epoch J2000.0 catalog data.

1) Units and the system of positions and proper motions

It is assumed that the positions and proper motions of the catalog to be transformed to J2000.0 are referred to the epoch and equinox of B1950.0. The positions are assumed to be in units of seconds of time in right ascension and seconds of arc in declination. However, when these quantities are used as arguments of trigonometric functions, it is left to the user to express them in degrees or radians as necessary. Similarly, the inverse of a trigonometric function is assumed to be degrees. The proper motions are assumed to be in units of seconds of time per tropical century in right ascension and seconds of arc per tropical century in declination. Radial velocities are taken to be in units of km/s and the parallaxes are in seconds of arc. It is also assumed that the catalog in question is referred to the system of the FK4. If not, steps must be taken either to refer the catalog to the FK4 system or to alter the appropriate steps in the discussion that follows.

2) Elliptic aberration

Correct the catalog right ascension and declination (α_{cat}, δ_{cat}) for the elliptic terms in aberration using Eqs. (5) and (6). If necessary, correct the catalog proper motions (μ_{cat},μ'_{cat}) for the elliptic terms in aberration using Eqs. (8) and (9).

3) Position and velocity vectors (B1950.0 to 1984 January $1^d\ 0^h$)

Form the position vector $\mathbf{u}_1$ and the velocity vector $\dot{\mathbf{u}}_1$ from the corrected positions α and δ, the proper motions μ and μ' (corrected if necessary), the parallax p, and the radial velocity $\dot{r}$. The position vector at the equinox and epoch of B1950.0, is

$$\mathbf{u}_1(t_0) = \begin{bmatrix} r\cos\delta\cos\alpha \\ r\cos\delta\sin\alpha \\ r\sin\delta \end{bmatrix}, \quad \text{where } r = 1/\sin p. \tag{25}$$

The velocity vector $\dot{\mathbf{u}}_1$, in rectilinear coordinates, at the equinox and epoch of B1950.0, is

$$\dot{\mathbf{u}}_1 = \begin{bmatrix} -\sin\alpha\cos\delta & -\cos\alpha\sin\delta & \cos\alpha\cos\delta \\ \cos\alpha\cos\delta & -\sin\alpha\sin\delta & \sin\alpha\cos\delta \\ 0 & \cos\delta & \sin\delta \end{bmatrix} \times \begin{bmatrix} 15r\mu K \\ r\mu' K \\ \dot{r}S \end{bmatrix}, \qquad (26)$$

where $S = 86400 \times 36524.2198782/(1.49597870 \times 10^8)$ is the conversion from km/s to AU per tropical century and K is 1/206264.806247,the conversion factor from seconds of arc to radians. Thus the unit of $\mathbf{u}_1$ is astronomical units (AU), and $\dot{\mathbf{u}}_1$ is in AU per tropical century. If p is not known, it may be set equal to any convenient value without altering the transformation of the angular information. We choose the value $p = 1''$ for this application, but $r = 1$ would have served equally well. The unit of r is AU. The position vector at epoch t_1, 1984 January $1^{\rm d}0^{\rm h}$ (2445700.5), but still referred to the equinox of B1950.0, is

$$\mathbf{u}_2(t_1) = \mathbf{u}_1(t_0) + \dot{\mathbf{u}}_1(t_1 - t_0), \qquad (27)$$

where $(t_1 - t_0)$ is the number of tropical centuries between epoch t_0 (B1950.0) and epoch t_1.

4) Precession from B1950.0 to 1984 January 1^d 0^h

Apply the precession from B1950.0 to 1984 January $1^{\rm d}0^{\rm h}$ to the vectors $\mathbf{u}_2(t_1)$ and $\dot{\mathbf{u}}_1$, using the precession matrix $\mathbf{P}_1$ calculated from the angles ζ_0, z, and θ given in Sec. II *e1*:

$$\mathbf{P}_1 = \begin{bmatrix} +0.999965667560 & -0.007599409538 & -0.003303433841 \\ +0.007599409535 & +0.999971123992 & -0.000012553023 \\ +0.003303433846 & -0.000012551554 & +0.999994543569 \end{bmatrix}. \qquad (28)$$

Then

$$\mathbf{u}_3(t_1) = \mathbf{P}_1 \mathbf{u}_2(t_1) \qquad (29)$$

and

$$\dot{\mathbf{u}}_2 = \mathbf{P}_1 \dot{\mathbf{u}}_1.$$

From the components x_3, y_3, z_3 and $\dot{x}_2, \dot{y}_2, \dot{z}_2$ of the vectors $\mathbf{u}_3$ and $\dot{\mathbf{u}}_2$, respectively, calculate the right ascension and declination and their proper motions at 1984 January $1^{\rm d}0^{\rm h}$:

$$\alpha_1 = \tan^{-1}(y_3/x_3),$$

$$\delta_1 = \tan^{-1}(z_3/\sqrt{x_3^2 + y_3^2}), \qquad (30)$$

$$\mu_2 = \frac{(x_3\dot{y}_2 - y_3\dot{x}_2)}{15K(x_3^2 + y_3^2)},$$

$$\mu_2' = \frac{r_3^2\dot{z}_2 - z_3(x_3\dot{x}_2 + y_3\dot{y}_2 + z_3\dot{z}_2)}{Kr_3^2\sqrt{r_3^2 - z_3^2}}, \qquad (31)$$

where

$$r_3^2 = x_3^2 + y_3^2 + z_3^2.$$

The equations for μ_2 and μ_2' follow directly from the total time derivative of the equations for α_1 and δ_1. The units of α_1 and δ_1 are degrees, while μ_2 and μ_2' are in seconds of time per tropical century and seconds of arc per tropical century, respectively, with K defined in Eq. (26).

The parallax p_3, at 1984 January $1^{\rm d}0^{\rm h}$, consistent with a model assuming linear space motion, is given by

$$p_3 = \sin^{-1}(1/r_3) \qquad (32)$$

and the new radial velocity in AU per tropical century, $\dot{r}_3$, is

$$\dot{r}_3 = (x_3\dot{x}_2 + y_3\dot{y}_2 + z_3\dot{z}_2)/r_3. \qquad (33)$$

5) Right ascension zero-point correction

Apply the zero-point correction to the right ascension:

$$\alpha_2 = \alpha_1 + 0\overset{s}{.}06390,$$

$$\delta_2 = \delta_1, \qquad (34)$$

where the right ascension and declination (α_1, δ_1) are expressed in seconds of time and seconds of arc, respectively.

6) Transformation of the proper-motion system

Applying the corrections and transformations discussed in Sec. II*f* to the proper motions:

$$\begin{aligned} \mu_3 &= (\mu_2 + 0.085)F - (m - m_0) \\ &\quad - (n \sin\alpha_2 - n_0 \sin\alpha_1)\tan\delta_2 \\ &= (\mu_2 + 0.085)F - (307.465950 - 307.396812) \\ &\quad - (133.629829 \sin\alpha_2 - 133.600750 \sin\alpha_1)\tan\delta_2 \end{aligned} \qquad (35)$$

$$\begin{aligned} \mu_3' &= \mu_2' F - (n\cos\alpha_2 - n_0\cos\alpha_1) \\ &= \mu_2' F - (2004.44743\cos\alpha_2 - 2004.01126\cos\alpha_1), \end{aligned}$$

where

$$F = 1.00002135903.$$

From Eq. (35), it becomes evident that a star precisely at or within a few seconds of arc of a pole must be treated as a special case. Historically, proper motions have been given in right ascension and declination without problems. If a star is very close to the pole, the singularity is avoided by considering the real displacement $\mu_3 \cos\delta_2$.

7) Position and velocity vectors (1984 January 1^d 0^h to J2000.0)

Form the vectors $\mathbf{u}_4$ and $\dot{\mathbf{u}}_4$ from α_2, δ_2, μ_3, and μ_3':

$$\mathbf{u}_4(t_1) = \begin{bmatrix} r_3\cos\delta_2\cos\alpha_2 \\ r_3\cos\delta_2\sin\alpha_2 \\ r_3\sin\delta_2 \end{bmatrix},$$

$$\dot{\mathbf{u}}_4 = \begin{bmatrix} -\sin\alpha_2\cos\delta_2 & -\cos\alpha_2\sin\delta_2 & \cos\alpha_2\cos\delta_2 \\ \cos\alpha_2\cos\delta_2 & -\sin\alpha_2\sin\delta_2 & \sin\alpha_2\cos\delta_2 \\ 0 & \cos\delta_2 & \sin\delta_2 \end{bmatrix} \times \begin{bmatrix} 15r_3\mu_3 K \\ r_3\mu_3' K \\ \dot{r}_3 F \end{bmatrix}, \qquad (36)$$

where the unit of $\mathbf{u}_4$ is astronomical units (AU), and $\dot{\mathbf{u}}_4$ is in AU per Julian century. Form the position vector $\mathbf{u}_5$ at epoch J2000.0:

$$\mathbf{u}_5(t_2) = \mathbf{u}_4(t_1) + \dot{\mathbf{u}}_4(t_2 - t_1), \tag{37}$$

where $(t_2 - t_1)$ is the number of Julian centuries between epoch t_2 (J2000.0) and epoch t_1, 1984 January $1^{\rm d}0^{\rm h}$.

8) Precession from 1984 January 1^d0^h to J2000.0

Apply the precession from 1984 January $1^{\rm d}0^{\rm h}$ to J2000.0 to the vectors $\mathbf{u}_5$ and $\dot{\mathbf{u}}_4$, using the following precession matrix calculated from the angles $\zeta_{\rm A}$, $z_{\rm A}$, and $\theta_{\rm A}$ given in Sec. II*e2*:

$$\mathbf{P}_2 = \begin{bmatrix} +0.999992390029 & -0.003577999042 & -0.001554929623 \\ +0.003577999042 & +0.999993598937 & -0.000002781855 \\ +0.001554929624 & -0.000002781702 & +0.999998791092 \end{bmatrix}, \tag{38}$$

$$\mathbf{u}_6 = \mathbf{P}_2\mathbf{u}_5(t_2),$$
$$\dot{\mathbf{u}}_5 = \mathbf{P}_2\dot{\mathbf{u}}_4. \tag{39}$$

From the components of the vectors $\mathbf{u}_6$ and $\dot{\mathbf{u}}_5$, the right ascension and declination and their proper motions at J2000.0 along with the modified parallax and radial velocity may be computed from a reapplication of Eqs. (30)–(33). The radial velocity will be in AU per Julian century and the conversion factor to km/s is $1.49597870\times10^8/(86400\times36525)$. If neither the radial velocity nor the parallax of a star were known at the beginning of this procedure, then the fictitious values produced by the procedure should be ignored. If only the parallax is known, then the original B1950.0 value should be brought forward without modification.

III. TRANSFORMATION OF OBSERVATIONAL CATALOGS

If the transformation is applied to an observational catalog in which no proper motions are given, and the mean positions are referred to the mean epoch of observation and the equinox of B1950.0, then the mean positions may be precessed to the equinox of the mean epoch of observation, which, in general, will be different for each star. The right ascension and declination should be corrected for terms in elliptic aberration corresponding to the mean epoch of observation, provided they had not already been removed when deriving the catalog place. Special care must be taken if the epochs of observation in right ascension and declination are different, which happens frequently in modern observational catalogs.

The right ascensions, if they were referred to the FK4 system, should be corrected for the zero-point error evaluated at the mean epoch of observation in right ascension. The positions may be brought forward to J2000.0 using the equations for the IAU 1976 precession angles $\zeta_{\rm A}$, $z_{\rm A}$, and $\theta_{\rm A}$ given by Eq. (15) evaluated for the time interval from the mean equinox and epoch of observation to the equinox of J2000.0. In this case, the proper-motion part of the computation may be ignored. The same is true of catalogs of radiointerferometrically determined positions of quasars and other objects presumed to be located at extragalactic distances. However, the transformation of very long baseline (VLBI) and connected element interferometer (CEI) catalog positions from the equinox of B1950.0 to J2000.0 is slightly more complicated than the transformation of the FK4 catalog because of the way the zero point in right ascension is established. This usually involves a systematic adjustment to all right ascensions. Most adjustments refer to the averaged optical and radio right ascension of 3C 273B given by Hazard *et al.* (1971). To the extent that this right ascension was referred to the FK4 system at its mean epoch of observation, it also requires a correction to right ascension by an amount that the FK4 equinox required at the same epoch.

Another factor that should be taken into consideration when transforming interferometrically observed positions from the B1950.0 basis to the J2000.0 is the intrinsically high accuracy of such a catalog. In order to preserve the high accuracy of such catalogs, the effects that have been discussed earlier, namely, the elliptic terms in aberration, the equinox correction, and the use of Newcomb's precession constant to refer observations to a fixed equinox should all be removed in a manner closely analogous to the operations that would have been employed in reducing the original observations at the epoch of observation had the improved constants been available at that time. In general, this means that catalog positions must be precessed back to the equinox of the mean epoch of observation using the same precession constants as the catalog compiler used, and, if necessary, corrected for the elliptic terms in aberration and equinox error to the FK4 system. The modified position is then precessed to the equinox of J2000.0 using the Lieske *et al.* precession. This procedure does not recognize 1984 as an epoch at which any special condition is to be imposed and best preserves without distortion the high precision of interferometrically determined positions.

IV. NUMERICAL EXAMPLES

Numerical examples are given in Table I for selected FK4 stars. The selection was made on the basis of proximity to the north and south poles, the magnitude of the proper motion, parallax, and radial velocity, and one star that crosses the equator between B1950.0 and J2000.0. For each star, the first line gives the B1950.0 on the FK4 system data and the second gives the transformed FK4 data for J2000.0. The transformed FK4 data given in Table I are not the same as the FK5 data resulting from the incorporation of new observational results produced since the publication of the FK4 catalog.

TABLE I. Numerical examples for some FK4 stars.

	α	δ	μ	μ'	Px	R.V. (km/s)
10	00ʰ 17ᵐ 28ˢ774	− 65°10′06″70	+ 27ˢ141	+ 116″74	0″134	+ 8.70
10	00ʰ 20ᵐ 04ˢ3100	− 64°52′29″332	+ 26ˢ8649	+ 116″285	0″1340	+ 8.74
119	03ʰ 17ᵐ 55ˢ847	− 43°15′35″74	+ 27ˢ827	+ 74″76	0″156	+ 86.80
119	03ʰ 19ᵐ 55ˢ6785	− 43°04′10″830	+ 27ˢ7694	+ 73″050	0″1559	+ 86.87
239	06ʰ 11ᵐ 43ˢ975	− 74°44′12″46	+ 3ˢ105	− 21″12	0″115	+ 35.00
239	06ʰ 10ᵐ 14ˢ5196	− 74°45′11″036	+ 3ˢ1310	− 21″304	0″1150	+ 35.00
538	14ʰ 36ᵐ 11ˢ250	− 60°37′48″85	− 49ˢ042	+ 71″20	0″751	− 22.20
538	14ʰ 39ᵐ 36ˢ1869	− 60°50′07″393	− 49ˢ5060	+ 69″934	0″7516	− 22.18
793	21ʰ 04ᵐ 39ˢ935	+ 38°29′59″10	+ 35ˢ227	+ 318″47	0″292	− 64.00
793	21ʰ 06ᵐ 54ˢ5901	+ 38°44′44″969	+ 35ˢ3528	+ 320″206	0″2923	− 63.89
907	01ʰ 48ᵐ 48ˢ784	+ 89°01′43″74	+ 18ˢ107	− 0″43	0″000	+ 0.00
907	02ʰ 31ᵐ 49ˢ8131	+ 89°15′50″661	+ 21ˢ7272	− 1″571	0″0000	+ 0.00
923	20ʰ 15ᵐ 03ˢ004	− 89°08′18″48	+ 11ˢ702	− 0″09	0″000	+ 0.00
923	21ʰ 08ᵐ 46ˢ0652	− 88°57′23″667	+ 8ˢ4469	+ 0″171	0″0000	+ 0.00
1307	11ʰ 50ᵐ 06ˢ172	+ 38°04′39″15	+ 33ˢ873	− 580″57	0″116	− 98.30
1307	11ʰ 52ᵐ 58ˢ7461	+ 37°43′07″456	+ 33ˢ7156	− 581″216	0″1161	− 97.81
1393	14ʰ 54ᵐ 59ˢ224	+ 00°01′58″08	+ 0ˢ411	− 2″73	0″000	+ 0.00
1393	14ʰ 57ᵐ 33ˢ2650	− 00°10′03″240	+ 0ˢ4273	− 2″402	0″0000	+ 0.00

Note added in proof: The procedure given in Sec. II*f* parallels that used in the formation of the FK5 catalog at the Astronomisches Rechen-Institut, Heidelberg. However, according to C. A. Murray (private communication), the correction to the FK4 proper-motion system in right ascension should be made as Standish (1982) has done it, at the equinox of B1950.0, for two reasons. First, it avoids a fictitious rotation produced in the Aoki *et al.* approach which is not found in the Standish approach. The two approaches idffer systematically by as much as 0″005 and 0″003 per century in positions and proper motions, respectively, in the sence of a rotation about the equinox axis. Second, Fricke's corrections to Newcomb's precession and the equinox motion were determined from a discussion of FK4 proper motions given for the reference frame at epoch and equinox B1950.0.

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MEAN AND APPARENT PLACE COMPUTATIONS IN THE NEW IAU SYSTEM. II. TRANSFORMATION OF MEAN STAR PLACES FROM FK4 B1950.0 TO FK5 J2000.0 USING MATRICES IN 6-SPACE

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ABSTRACT

A 6×6 matrix method for transforming a catalog mean place from epoch and equinox B1950.0 on the FK4 system to epoch and equinox J2000.0 on the FK5 system is described. A step-by-step comparison is made between the matrix method and the classical spherical formulas.

I. INTRODUCTION

The main purposes of this paper are (1) to describe a 6×6 matrix method for transforming mean star places from the standard epoch and equinox of B1950.0 on the FK4 system to the standard epoch and equinox of J2000.0 on the FK5 system, and (2) to make a step-by-step comparison between this matrix method and the classical method described in Paper I.

The transformation from B1950.0 to J2000.0 is described in the next section. It is straightforward to express this transformation in the matrix notation adopted by Standish (1982) with the modifications communicated to Standish by Aoki and Soma (1983). There are two main advantages in the matrix method. First, if there are any changes in the transformation, such as a change in one of the parameters or a change in the order of operations, then it is easily incorporated into the matrix transformation. Second, it is easier to translate the matrix method into a high-level programming language such as FORTRAN or BASIC, and the program should run more efficiently. Third, there are no problems with singularities at the poles. Moreover, it is much easier to modify the algorithm if it is decided to change the transformation at a later stage (e.g., the alternative procedure suggested by C. A. Murray. See note added in proof to Paper I).

The various stages of the matrix transformation are discussed in more detail in Secs. IV, V, VII, IX, and X. In Sec. XI the complete matrix transformation is compared with the classical method. In most cases, the comparison is obvious. The exception is at the step in the transformation at which the proper motions are changed from the FK4 system to the FK5 system. In the matrix method, this transformation is implicit, while in the classical method it is explicit. It requires considerable algebraic manipulation to obtain explicit expressions from the matrix method which may then be compared with the classical expressions given in Paper I. When numerical comparisons are made between the two methods, they will agree to a precision of $\pm 0\rlap{.}''001$. The largest difference comes from the spherical equations for the E terms of aberration, partly from the corrections to position, but mainly from the corrections to the proper motions in right ascension and declination. Ignoring the effect of the E terms of aberration, the agreement is better than $\pm 2'' \times 10^{-10}$, provided the terms of order $\pm 1'' \times 10^{-7}$ involving l in the proper-motion equations are included in the classical case and all variables are calculated to the same precision in both methods. This includes the equinox correction, the precession angles, and m and n, the rates of change of the precession angles. In fact, the classical method calculates m and n at 1984 January 1.0 directly from their polynomial expressions, while in the 6×6 matrix method they are calculated at B1950.0 and J2000.0, and their values at 1984 January 1.0 are implicit. This produces differences between the two methods of about $\pm 3'' \times 10^{-6}$ in position and $\pm 2'' \times 10^{-5}$ per century in proper motion.

II. THE TRANSFORMATION FROM B1950.0 TO J2000.0

There has been much controversy in the literature over the correct procedure for the transformation (Aoki *et al.* 1983). The recommended transformation is given in Paper I. The transformation described here follows that recommendation. It should be noted that when transferring individual observations, as opposed to a catalog mean place, the safest method is to transform the observation back to the epoch of the observation, on the FK4 system (or in the system that was used to produce the observed mean place), convert to the FK5 system, and transform to the epoch and equinox J2000.0.

The transformation for a fundamental catalog position is as follows:

Step 1. Form the position and velocity vector from the FK4 catalog position, i.e., form the position and velocity vector from the right ascension, declination, proper motions, parallax, and radial velocity.

Step 2. Remove the E terms of aberration from the B1950.0 catalog mean place. There is also a question whether or not the E terms should be removed from the proper motions. The problem is discussed more fully in Paper I. If the FK4 catalog is used, they certainly do not have to be removed from stars within 10° of the poles because they have not been included (Lederle 1984).

Step 3. Apply space motion to the position vector to the epoch 1984 January 1.0, which is the epoch at which the sidereal time expression in terms of UT is changed (IAU 1977). This is an example of where one of the parameters of the transformation could be changed.

Step 4. Precess, using the FK4 precession constants, the position and velocity from B1950.0 to 1984 January 1.0.

Step 5. Apply the equinox correction FK4 to FK5 at 1984 January 1.0.

Step 6. Convert the proper motions from seconds of arc per tropical century to seconds of arc per Julian century.

Step 7. Precess, using the FK5 precession constants, the position and velocity from 1984 January 1.0 to J2000.0.

Step 8. Apply space motion to the position vector from

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1984 January 1.0 to J2000.0.

Step 9. Convert the position and velocity vector back to right ascension and declination and proper motions in right ascension and declination, and extract the parallax and radial velocity in the FK5 system.

III. NOTATION AND DEFINITIONS

α—right ascension in degrees.

δ—declination in degrees.

μ—proper motion in right ascension in arcseconds per century. On the FK4 system, tropical centuries are used, while Julian centuries are used on the FK5 system.

μ'—proper motion in declination in arcseconds per century.

π—parallax in arcseconds.

V—radial velocity in km s^{-1}.

C—the length of the century in days. Julian centuries C_J consist of 36 525 days, while tropical centuries $C_B = 36524.21987817305$ days.

$\mathscr{E}$—Epoch, e.g., 1984 January 1.0.

JD(date)—The Julian date, which is a function of the date or epoch. Julian dates for some relevant epochs are JD(B1950.0) = 2433282.42345905, JD(1984 January 1.0) = 2445700.5, and JD(J2000.0) = 2451545.0.

k—Conversion from km s^{-1} to AU per century. $k = 86400C/1.49597870\times 10^8$.

P—the precession matrix. $\mathbf{P}_n$, $\mathbf{P}_o$ are the precession matrices on the new FK5 and the old FK4 systems, respectively. $\mathbf{P}^{-1}$ and $\dot{\mathbf{P}}$ represent the inverse and the differential precession matrix. Whenever it occurs in this paper, $\dot{\mathbf{P}}^{-1}$ means $(\dot{\mathbf{P}})^{-1}$.

r—a column vector of position, where the transpose $\mathbf{r}' = (x, y, z)$.

$\dot{\mathbf{r}}$—a column vector of velocity, where the transpose $\dot{\mathbf{r}}' = (\dot{x}, \dot{y}, \dot{z})$.

$\mathbf{R}_i(\phi)$—the standard orthonormal rotation matrices $\mathbf{R}_i(\phi)$, $i = 1,2,3$, which rotate a right-handed set of axes x, y, z through an angle ϕ anticlockwise about the ith axis.

v—a vector in 6-space, where the transpose $\mathbf{v}' = (\mathbf{r}', \dot{\mathbf{r}}') = (x, y, z, \dot{x}, \dot{y}, \dot{z})$.

The rotations about the x, y, and z axes are represented by the following matrices

$$\mathbf{R}_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix},$$

$$\mathbf{R}_2(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix},$$

$$\mathbf{R}_3(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These matrices are orthonormal and therefore have the following properties:

$$\mathbf{R}_i^{\mathrm{T}}(\phi)\mathbf{R}_i(\phi) = \mathbf{I} = \mathbf{R}_i(\phi)\mathbf{R}_i^{\mathrm{T}}(\phi),$$

$$\mathbf{R}_i^{\mathrm{T}}(\phi)\mathbf{R}_j(\phi) = \mathbf{O}, \quad i \neq j,$$

where $\mathbf{R}_i^{\mathrm{T}}$ is the transpose of $\mathbf{R}_i$ and equals the inverse $\mathbf{R}_i^{-1}$. **I** is the unit matrix and **O** is the zero (null) matrix. Matrix inversion is very efficient for orthonormal matrices as it is just a matter of exchanging rows for columns. In all applications in this paper, ϕ is a function of time; thus the derivatives of the rotation matrices are given by

$$\dot{\mathbf{R}}_i(\phi) = \dot{\phi}\frac{\partial}{\partial\phi}\mathbf{R}_i(\phi), \quad i = 1,2,3,$$

and

$$\frac{\partial}{\partial\phi}\mathbf{R}_1(\phi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix},$$

$$\frac{\partial}{\partial\phi}\mathbf{R}_2(\phi) = \begin{bmatrix} -\sin\phi & 0 & -\cos\phi \\ 0 & 0 & 0 \\ \cos\phi & 0 & -\sin\phi \end{bmatrix},$$

$$\frac{\partial}{\partial\phi}\mathbf{R}_3(\phi) = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using the above definitions for the two 3×3 matrices $\mathbf{R}_i(\phi)$ and $\dot{\mathbf{R}}_i(\phi)$, we now need to work in 6-space and define a 6×6 matrix

$$\mathbf{Q}_i(\phi,\dot{\phi}) = \begin{bmatrix} \mathbf{R}_i(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i(\phi) & \mathbf{R}_i(\phi) \end{bmatrix}.$$

The reason for introducing **Q** and working in 6-space is as follows. Consider a position vector $\mathbf{r}_0$ which is rotated about the ith axis to produce a new vector $\mathbf{r}_1$, then

$$\mathbf{r}_1 = \mathbf{R}_i(\phi)\mathbf{r}_0.$$

Differentiating this equation with respect to time gives the relation between the two velocity vectors $\dot{\mathbf{r}}_0$, $\dot{\mathbf{r}}_1$ as follows:

$$\dot{\mathbf{r}}_1 = \dot{\phi}\frac{\partial}{\partial\phi}\mathbf{R}_i(\phi)\mathbf{r}_0 + \mathbf{R}_i(\phi)\dot{\mathbf{r}}_0.$$

Hence

$$\mathbf{v}_1 = \begin{bmatrix} \mathbf{R}_i(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i(\phi) & \mathbf{R}_i(\phi) \end{bmatrix} \begin{bmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \end{bmatrix} = \mathbf{Q}_i(\phi,\dot{\phi})\mathbf{v}_0.$$

In astronomy, the higher-order derivatives $\ddot{\mathbf{r}}$, etc., are negligible and therefore these two sets of equations in three dimensions are all that are required, and it is found to be more efficient to use one set of equations in 6-space.

The following properties for the **Q** matrix will be required later and are easily verified by substituting the above expressions for $\mathbf{R}_i(\phi)$ and $\partial\mathbf{R}_i(\phi)/\partial\phi$:

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi})\mathbf{Q}_i(\phi,\dot{\phi}) = \mathbf{I} = \mathbf{Q}_i(\phi,\dot{\phi})\mathbf{Q}_i^{-1}(\phi,\dot{\phi}),$$

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi}) = \mathbf{Q}_i(-\phi,-\dot{\phi}),$$

$$\mathbf{Q}_i^{-1}(\phi,\dot{\phi}) = \begin{bmatrix} R_i^{\mathrm{T}}(\phi) & \mathbf{O} \\ \dot{\phi}\dfrac{\partial}{\partial\phi}\mathbf{R}_i^{\mathrm{T}}(\phi) & \mathbf{R}_i^{\mathrm{T}}(\phi) \end{bmatrix}.$$

IV. CONVERSION FROM SPHERICAL TO VECTOR COORDINATES

In either method, matrix or classical, it is necessary at some stage of the calculation to convert from spherical to rectangular coordinates of position and velocity. Given that a star has position (α,δ) in degrees, proper motions (μ,μ') in seconds of arc per century, parallax (π) in seconds of arc, and radial velocity V in km s^{-1}, then the direction cosines of the position vector **r** and velocity vector $\dot{\mathbf{r}}$ in arcseconds per century are

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{bmatrix}$$

and

$$\dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\mu\sin\alpha\cos\delta - \mu'\cos\alpha\sin\delta \\ +\mu\cos\alpha\cos\delta - \mu'\sin\alpha\sin\delta \\ \mu'\cos\delta \end{bmatrix} + kV\pi\mathbf{r},$$

where $k = 86400C/(1.49597870\times 10)^8$ and C is the length of the century in days. The units of $\dot{\mathbf{r}}$ are arcseconds per century, Julian or tropical as appropriate, and $\mathbf{r}$ is automatically a unit vector.

Conversely, it is also necessary at some stage of the calculation to convert back from rectangular coordinates of position and velocity to spherical coordinates. It may no longer be the case that $\mathbf{r}$ is a unit vector. Hence given the vectors $\mathbf{r}' = (x, y, z)$ and $\dot{\mathbf{r}}' = (\dot{x}, \dot{y}, \dot{z})$, where $\dot{\mathbf{r}}'$ is in arcseconds per century, then the right ascension and declination (α,δ) and proper motions (μ,μ') in arcseconds per century are obtained from

$$\cos\alpha\cos\delta = x/r, \quad \sin\alpha\cos\delta = y/r, \quad \sin\delta = z/r,$$

$$\mu = \frac{x\dot{y} - y\dot{x}}{x^2+y^2}, \quad \mu' = \frac{\dot{z}(x^2+y^2) - z(x\dot{x}+y\dot{y})}{r^2\sqrt{x^2+y^2}},$$

where

$$r = \sqrt{x^2+y^2+z^2}.$$

In the conversion from B1950.0 to J2000.0, the parallax and velocity at J2000.0 (π_1, V_1) can be obtained from $\mathbf{r}$ and $\dot{\mathbf{r}}$ together with the parallax (π) and velocity (V) at B1950.0; thus

$$\pi_1 = \pi/r, \quad V_1 = (x\dot{x} + y\dot{y} + z\dot{z})/k\pi r.$$

However, if $\pi = 0$, then $V_1 = V$.

V. THE REMOVAL OF THE ELLIPTIC TERMS OF ABERRATION

The equatorial velocity components of the E terms of aberration (Emerson 1973) referred to the equinox of date are given by

$$\mathbf{B} = \begin{bmatrix} -\Delta D \\ +\Delta C \\ +\Delta C\tan\epsilon \end{bmatrix}.$$

In a fixed frame at B1950.0, the components at date are given by

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}, \tag{1}$$

where $\mathbf{P}$ is the precession matrix from B1950.0 to date. Provided $\mathbf{A}$ is in the appropriate units, then the direction to the star $\mathbf{r}_0$, corrected for the E terms of aberration, is given by

$$\mathbf{r}_0 = \mathbf{r}_{\text{cat}} - [\mathbf{A} - (\mathbf{r}'_{\text{cat}}\ \mathbf{A})\mathbf{r}_{\text{cat}}], \tag{2}$$

where $\mathbf{r}_{\text{cat}}$ is the catalog position vector, $[\mathbf{A} - (\mathbf{r}'_{\text{cat}}\ \mathbf{A})\mathbf{r}_{\text{cat}}]$ is the component of $\mathbf{A}$ in the direction perpendicular to $\mathbf{r}_{\text{cat}}$, since $\mathbf{r}'_{\text{cat}}$ is the transpose of $\mathbf{r}_{\text{cat}}$, and $(\mathbf{r}'_{\text{cat}}\ \mathbf{A})$ is the scalar product. The numerical values of the elements of $\mathbf{A}$ at B1950.0 when $\mathbf{P} = \mathbf{I}$ are

$$\mathbf{A} = \begin{bmatrix} -1.62557 \\ -0.31919 \\ -0.13843 \end{bmatrix} \times 10^{-6} \text{ radians}$$

using the expressions for ΔC and ΔD given in Paper I, Sec. II *d 1*, Eq. (7). The classical formula (Paper I, Sec. II *d 1*, Eqs. (5) and (6)) may be derived from this expression using a first-order approximation.

Aoki (1983) has pointed out that the E terms of aberration may have affected the proper motions as well. In vector notation, the effect is derived by differentiating Eq. (2) as follows:

$$\dot{\mathbf{r}}_0 = \dot{\mathbf{r}}_{\text{cat}} - \dot{\mathbf{A}} + (\mathbf{r}'_{\text{cat}}\ \dot{\mathbf{A}})\mathbf{r}_{\text{cat}} + (\dot{\mathbf{r}}'_{\text{cat}}\ \mathbf{A})\mathbf{r}_{\text{cat}} + (\mathbf{r}'_{\text{cat}}\ \mathbf{A})\dot{\mathbf{r}}_{\text{cat}}, \tag{3}$$

where the terms $(\dot{\mathbf{r}}'_{\text{cat}}\ \mathbf{A})\mathbf{r}_{\text{cat}}$ and $(\mathbf{r}'_{\text{cat}}\ \mathbf{A})\dot{\mathbf{r}}_{\text{cat}}$ are very small and may be neglected unless a precision better than $\pm 1'' \times 10^{-3}$ is required.

In the fixed frame at B1950.0, the expression for $\dot{\mathbf{A}}$ at date is given by differentiating Eq. (1); thus

$$\dot{\mathbf{A}} = \dot{\mathbf{P}}^{-1}\mathbf{B} + \mathbf{P}^{-1}\dot{\mathbf{B}},$$

where

$$\dot{\mathbf{P}}^{-1} = \begin{bmatrix} 0 & \dot{\zeta}_A + \dot{z}_A & \dot{\theta}_A \\ -\dot{\zeta}_A - \dot{z}_A & 0 & 0 \\ -\dot{\theta}_A & 0 & 0 \end{bmatrix}$$

and ζ_A, z_A, and θ_A are the precession angles. At B1950.0, differentiating Andoyer's expressions for the precession angles which are given in Sec. VIII on the FK4 system, with $T = 1$ and $t = 0$, gives

$$\dot{\zeta}_A + \dot{z}_A = 4609''.90 \text{ per tropical century},$$

$$\dot{\theta}_A = 2004''.26 \text{ per tropical century}.$$

Also, $\mathbf{P}^{-1} = \mathbf{I}$ and

$$\dot{\mathbf{B}} = \begin{bmatrix} -\Delta\dot{D} \\ +\Delta\dot{C} \\ \Delta\dot{C}\tan\epsilon + \Delta C\dot{\epsilon}_r/\cos^2\epsilon \end{bmatrix} = \begin{bmatrix} +2''.9941 \\ -9''.0738 \\ -3''.9174 \end{bmatrix} \times 10^{-3} \text{ per tropical century}$$

using the expressions for $\Delta\dot{C}$ and $\Delta\dot{D}$ given in Paper I, Sec. II *d2*, Eqs. (10). Hence

$$\dot{\mathbf{A}} = \begin{bmatrix} +1''.245 \\ -1''.580 \\ -0''.659 \end{bmatrix} \times 10^{-3} \text{ per tropical century}.$$

The classical formula for the correction to the proper motions for the E terms is equivalent to Eq. (3) to first order. In the case where the E terms have been allowed for when deriving the catalog proper motions, the procedure in the vector method has to follow the classical method more closely. After the position vector $\mathbf{r}_{\text{cat}}$ has been corrected for the E terms using Eq. (2), the corrected right ascension and declination of the position vector $\mathbf{r}_0$ have to be determined and they are used to form the velocity vector.

VI. THE PRECESSION MATRIX IN 3-SPACE

The precession matrix $\mathbf{P}$, which precesses equatorial rectangular coordinates from a fixed equinox and equator $\mathscr{E}_F$ to one of date $\mathscr{E}_D$ in 3-space, is given by

$$\mathbf{P}[\mathscr{E}_F, \mathscr{E}_D] = \mathbf{R}_3(-z_A)\mathbf{R}_2(+\theta_A)\mathbf{R}_3(-\zeta_A),$$

where ζ_A, z_A, θ_A are the precession angles, which are evaluated using the appropriate time arguments. The expressions for the precession angles in seconds of arc are given in Sec. VIII. The precession matrix is made up of three rotations, applied in an appropriate order. The first rotation is through the angle $-\zeta_A$ about the z axis, the second rotation is through the angle $+\theta_A$ about the y axis, and the final rotation is through the angle $-z_A$ about the z axis.

Having calculated the precession angles for the matrix **P**, the inverse matrix $\mathbf{P}^{-1}$ can be calculated in various ways, for example,

$$\begin{aligned}\mathbf{P}^{-1} &= \mathbf{R}_3^{-1}(-\zeta_A)\mathbf{R}_2^{-1}(+\theta_A)\mathbf{R}_3^{-1}(-z_A)\\ &= \mathbf{R}_3^{T}(-\zeta_A)\mathbf{R}_2^{T}(+\theta_A)\mathbf{R}_3^{T}(-z_A)\\ &= \mathbf{R}_3(+\zeta_A)\mathbf{R}_2(-\theta_A)\mathbf{R}_3(+z_A).\end{aligned}$$

Alternatively, the precession angles may be recalculated since $\mathbf{P}^{-1}[\mathscr{E}_F,\mathscr{E}_D] = \mathbf{P}[\mathscr{E}_D,\mathscr{E}_F]$. Differentiating **P**, we find that the differential precession matrix is given by

$$\begin{aligned}\dot{\mathbf{P}} &= \dot{\mathbf{R}}_3(-z_A)\mathbf{R}_2(+\theta_A)\mathbf{R}_3(-\zeta_A)\\ &+ \mathbf{R}_3(-z_A)\dot{\mathbf{R}}_2(+\theta_A)\mathbf{R}_3(-\zeta_A)\\ &+ \mathbf{R}_3(-z_A)\mathbf{R}_2(+\theta_A)\dot{\mathbf{R}}_3(-\zeta_A),\end{aligned}$$

where $\dot{\mathbf{R}}$ is defined in Sec. III. $\dot{\mathbf{P}}$ is required when comparing the two methods (Sec. XI).

VII. THE PRECESSION MATRIX IN 6-SPACE

In 6-space, the notation for the precession matrix has to be modified to include a further parameter s, where $s = 0$ when precessing from one inertial frame to another inertial frame and $s = 1$ when precessing from an inertial frame to a non-inertial frame (i.e., rotating frame of date). The expression for **P** becomes

$$\begin{aligned}\mathbf{P}[\mathscr{E}_F,\mathscr{E}_D,s] &= \begin{bmatrix}\mathbf{P} & \mathbf{O}\\ s\dot{\mathbf{P}} & \mathbf{P}\end{bmatrix}\\ &= \mathbf{Q}_3(-z_A,-s\dot{z}_A)\mathbf{Q}_2(+\theta_A,+s\dot{\theta}_A)\\ &\times\mathbf{Q}_3(-\zeta_A,-s\dot{\zeta}_A).\end{aligned}$$

For the conversion from FK4 B1950.0 to FK5 J2000.0 in Sec.X, the case $s = 1$ is required. The inverse is also required with $s = 1$, and is given by

$$\begin{aligned}\mathbf{P}^{-1}[\mathscr{E}_F,\mathscr{E}_D,1] &= \begin{bmatrix}\mathbf{P} & \mathbf{O}\\ \dot{\mathbf{P}} & \mathbf{P}\end{bmatrix}^{-1} = \begin{bmatrix}\mathbf{P}^{-1} & \mathbf{O}\\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1}\end{bmatrix}\\ &= [\mathbf{Q}_3(-z_A,-\dot{z}_A)\mathbf{Q}_2(+\theta,+\dot{\theta}_A)\\ &\times\mathbf{Q}_3(-\zeta,-\dot{\zeta}_A)]^{-1}.\end{aligned}$$

VIII. NUMERICAL EXPRESSIONS FOR THE PRECESSION ANGLES AND THEIR RATES OF CHANGE

The equatorial precession angles ζ_A, z_A, θ_A are given as polynomial functions of T and t. T transforms the equations from the basic epoch $\mathscr{E}_0$ (e.g., B1850.0 or J2000.0) to the required fixed (initial) epoch $\mathscr{E}_F$ (e.g., B1950.0), while t transforms the equations from the fixed (initial) epoch $\mathscr{E}_F$ (e.g., B1950.0 or J2000.0) to the epoch of date $\mathscr{E}_D$.

$$\zeta_A = \zeta_A(T,t),\quad z_A = z_A(T,t),\quad \theta_A = \theta_A(T,t),$$

where

$$T = [\mathrm{JD}(\mathscr{E}_F) - \mathrm{JD}(\mathscr{E}_0)]/C$$

and

$$t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C,$$

with C the number of days in the century, Julian or tropical as appropriate. The conversion of mean star places from B1950.0 on the FK4 system to J2000.0 on the FK5 system requires not only the precession angles but also their rates of change with respect to time to be defined on both systems. The precession rates are given by

$$\dot{\zeta}_A = \frac{d}{dt}\zeta_A(T,t),\quad \dot{z}_A = \frac{d}{dt}z_A(T,t),\quad \dot{\theta}_A = \frac{d}{dt}\theta_A(T,t).$$

There is still some argument as to whose (i.e., Newcomb, or Andoyer, or Kinoshita) definitions of precession to use with the FK4 system. In this paper, we have used Andoyer's, and in these equations the basic epoch is $\mathscr{E}_0 = $ B1850.0, and the time arguments are fractions of a tropical century.

$$\begin{aligned}\zeta_A &= (2303''.5545 + 1''.39720T + 0''.000060T^2)t + (0''.30240 - 0''.000270T)t^2 + 0''.017995t^3,\\ z_A &= (2303''.5545 + 1''.39720T + 0''.000060T^2)t + (1''.09480 + 0''.000390T)t^2 + 0''.018325t^3,\\ \theta_A &= (2005''.112 - 0''.8529T - 0''.00037T^2)t + (-0''.4265 - 0''.00037T)t^2 - 0''.04180t^3.\end{aligned}$$

In this application, $T = 1$ as the fixed epoch is $\mathscr{E}_F = $ B1950.0, and $t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C_B$. The precession angles for the FK5 system have been defined by Lieske (1979) and adopted by the IAU. In these equations, the basic epoch of the equations is $\mathscr{E}_0 = $ J2000.0 or $\mathrm{JD}(\mathscr{E}_0) = 2451545.0$, and the time arguments are fractions of a Julian century.

$$\begin{aligned}\zeta_A &= (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (0''.30188 - 0''.000344T)t^2 + 0''.017998t^3,\\ z_A &= (2306''.2181 + 1''.39656T - 0''.000139T^2)t + (1''.09468 + 0''.000066T)t^2 + 0''.018203t^3,\\ \theta_A &= (2004''.3109 - 0''.85330T - 0''.000217T^2)t + (-0''.42665 - 0''.000217T)t^2 - 0''.041833t^3.\end{aligned}$$

In this application, $T = [\mathrm{JD}(\mathscr{E}_F) - \mathrm{JD}(\mathscr{E}_0)]/C_J$ and $t = [\mathrm{JD}(\mathscr{E}_D) - \mathrm{JD}(\mathscr{E}_F)]/C_J$. In both systems, the precession angles are subscripted by the symbol A, which indicates that the precession angles are accumulated, and the rates of change are expressed in seconds of arc per century.

IX. EQUINOX CORRECTION

Fricke (1982) has determined that the FK4 right ascension system requires a correction of $+0''.525$ at B1950.0, and in general at epoch $\mathscr{E}$ the correction to right ascension should be

$$E_{\mathscr{E}} = E_{50} + \dot{E}\,[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{B1950.0})]/C_B,$$

where

$$E_{50} = 0''.525 \text{ and } \dot{E} = 1''.275.$$

In the classical method, this correction is applied separately to the right ascensions and proper motions in right ascension. In the vector method, the correction is applied by means of a rotation about the z axis. The total effect on both the position and velocity vectors is given by $\mathbf{Q}_3(-E_{\mathscr{E}}, -\dot{E})$, where $E_{\mathscr{E}}$ is expressed in degrees and $\dot{E}$ in arcseconds per tropical century.

X. THE COMPLETE TRANSFORMATION IN MATRIX NOTATION

The six steps 3, 4, 5, 6, 7, and 8 in Sec. II may be represented by successive multiplication by the six matrices M_1, M_2, M_3, M_4, M_5, and M_6 on the position and velocity vector $\mathbf{v}_0$ at B1950.0 producing the position and velocity vector $\mathbf{v}_1$ at J2000.0, where

$$\mathbf{v}_1 = \mathsf{M}_6\mathsf{M}_5\mathsf{M}_4\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1\mathbf{v}_0 \tag{4}$$

and the six matrices are defined as follows:

M_1—Adds space motion between the standard epoch B1950.0 and $\mathscr{E}$ to the position vector at B1950.0.

$$\mathsf{M}_1 = \begin{bmatrix} \mathbf{I} & t_0\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix},$$

where $t_0 = c[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{B}1950.0)]/C_B$ and $c = \pi/(180\times 3600)$ is a factor that converts seconds of arc to radians.

M_2—Applies FK4 precession from B1950.0 to $\mathscr{E}$, to the position and velocity in 6-space.

$$\mathsf{M}_2 = \mathbf{P}_o[\mathrm{B}1950.0,\mathscr{E},1]$$

$$= \mathbf{Q}_3(-z_A, -\dot{z}_A)Q_2(\theta_A,\dot{\theta}_A)Q_3(-\zeta_A, -\dot{\zeta}_A).$$

M_3—Adds the equinox correction to the right ascension at epoch $\mathscr{E}$.

$$\mathsf{M}_3 = \mathbf{Q}_3(-E_{\mathscr{E}}, -\dot{E}),$$

where $E_{\mathscr{E}} = E_{50} + \dot{E}[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{B}1950.0)]/C_B$ and $E_{50} = 0\rlap{.}''525$ and $\dot{E} = 1\rlap{.}''275$.

M_4—Converts the proper motions from tropical centuries to Julian centuries.

$$\mathsf{M}_4 = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & F\mathbf{I} \end{bmatrix},$$

where $F = C_J/C_B$.

M_5—Applies FK5 precession from $\mathscr{E}$ to J2000.0, to the position and velocity in 6-space.

$$\mathsf{M}_5 = \mathbf{P}_n^{-1}[\mathrm{J}2000.0,\mathscr{E},1]$$

$$= \mathbf{Q}_3(+\zeta_A, +\dot{\zeta}_A)\mathbf{Q}_2(-\theta_A, -\dot{\theta}_A)$$

$$\times\mathbf{Q}_3(+z_A, +\dot{z}_A)$$

M_6—Adds space motion between $\mathscr{E}$ and J2000.0 to the position vector at $\mathscr{E}$.

$$\mathsf{M}_6 = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix},$$

where $t_1 = c[\mathrm{JD}(\mathscr{E}) - \mathrm{JD}(\mathrm{J}2000.0)]/C_J$ and $c = \pi/(180\times 3600)$.

The product of the six matrices may be represented by the single matrix

$$\mathsf{M} = \mathsf{M}_6\mathsf{M}_5\mathsf{M}_4\mathsf{M}_3\mathsf{M}_2\mathsf{M}_1.$$

In particular, when the epoch $\mathscr{E}$ = 1984 January 1.0, the numerical expression for M, printed to 15 decimal places, is

$$\begin{bmatrix}
0.999925678186902 & -0.011182059642247 & -0.004857946558960 & 0.000002423950176 & -0.000000027106627 & -0.000000011776558 \\
0.011182059571766 & 0.999937478448132 & -0.000027176441185 & 0.000000027106627 & 0.000002423978783 & -0.000000000065874 \\
0.004857946721186 & -0.000027147426498 & 0.999988199738770 & 0.000000011776559 & -0.000000000065816 & 0.000002424101735 \\
-0.000541652366951 & -0.237968129744288 & 0.436227555856097 & 0.999947035154614 & -0.011182506121805 & -0.004857669684959 \\
0.237917612131583 & -0.002660763319071 & -0.008537771074048 & 0.011182506007242 & 0.999958833818833 & -0.000027184471371 \\
-0.436111276039270 & 0.012259092261564 & 0.002119110818172 & 0.004857669948650 & -0.000027137309539 & 1.000009560363559
\end{bmatrix}$$

XI. THE COMPARISON

This section compares the method that uses matrices in six space with the classical method in three space. Equation (4) may be written with $\mathbf{P} = \mathbf{P}_n$ and $E = E_{\mathscr{E}}$:

$$\begin{pmatrix} \mathbf{r}_1 \\ \dot{\mathbf{r}}_1 \end{pmatrix} = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & F\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}_3(-E) & \mathbf{O} \\ \dot{\mathbf{R}}_3(-E) & \mathbf{R}_3(-E) \end{bmatrix} \begin{bmatrix} \mathbf{P}_o & \mathbf{O} \\ \dot{\mathbf{P}}_o & \mathbf{P}_o \end{bmatrix} \begin{bmatrix} \mathbf{I} & t_0\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{r}_0 \\ \dot{\mathbf{r}}_0 \end{pmatrix}.$$

The transformation from $\mathscr{E}$ to J2000.0 on the FK5 system is

$$\begin{pmatrix} \mathbf{r}_1 \\ \dot{\mathbf{r}}_1 \end{pmatrix} = \mathsf{M}_6\mathsf{M}_5\mathbf{v}^5_{\mathscr{E}} = \begin{bmatrix} \mathbf{I} & -t_1\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{O} \\ \dot{\mathbf{P}}^{-1} & \mathbf{P}^{-1} \end{bmatrix} \begin{pmatrix} \mathbf{r}^5_{\mathscr{E}} \\ \dot{\mathbf{r}}^5_{\mathscr{E}} \end{pmatrix}. \tag{5}$$

Hence at $\mathscr{E}$ the middle part of the transformation from FK4 to FK5 is given by

$$\begin{pmatrix} \mathbf{r}^5_{\mathscr{E}} \\ \dot{\mathbf{r}}^5_{\mathscr{E}} \end{pmatrix} = \mathsf{M}_4\mathsf{M}_3\mathbf{v}^4_{\mathscr{E}} = \begin{pmatrix} \mathbf{R}_3(-E)\mathbf{r}^4_{\mathscr{E}} \\ F[\dot{\mathbf{R}}_3(-E)\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\dot{\mathbf{r}}^4_{\mathscr{E}}] \end{pmatrix}. \tag{6}$$

The transformation from B1950.0 to $\mathscr{E}$ on the FK4 system is

$$\begin{pmatrix} \mathbf{r}^4_{\mathscr{E}} \\ \dot{\mathbf{r}}^4_{\mathscr{E}} \end{pmatrix} = \mathsf{M}_2\mathsf{M}_1\mathbf{v}_0 = \begin{pmatrix} \mathbf{P}_o(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) \\ \dot{\mathbf{P}}_o(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) + \mathbf{P}_o\dot{\mathbf{r}}_0 \end{pmatrix}. \tag{7}$$

Consider the space part of the transformation from Eq. (5),

$$\mathbf{r}_1 = \mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} - t_1(\mathbf{P}^{-1}\dot{\mathbf{r}}^5_{\mathscr{E}} + \dot{\mathbf{P}}^{-1}\mathbf{r}^5_{\mathscr{E}}).$$

From the velocity part of Eq. (5)

$$\dot{\mathbf{r}}_1 = \mathbf{P}^{-1}\dot{\mathbf{r}}^5_{\mathscr{E}} + \dot{\mathbf{P}}^{-1}\mathbf{r}^5_{\mathscr{E}}.$$

Hence

$$\mathbf{r}_1 = \mathbf{P}^{-1}(\mathbf{r}^5_{\mathscr{E}} - t_1\mathbf{P}\dot{\mathbf{r}}_1).$$

Using Eqs. (6) and (7) to express $\mathbf{r}^5_{\mathscr{E}}$ in terms of $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, the position at J2000.0 can be written in terms that are directly comparable with the classical method. Thus

$$\mathbf{r}_1 = \mathbf{P}^{-1}[\mathbf{R}_3(-E)\mathbf{P}_{\rm o}(\mathbf{r}_0 + t_0\dot{\mathbf{r}}_0) - t_1\mathbf{P}\dot{\mathbf{r}}_1].$$

To complete the comparison, an expression connecting $\dot{\mathbf{r}}^5_{\mathscr{E}}$ with $\dot{\mathbf{r}}^4_{\mathscr{E}}$ is required. This is obtained by considering the velocity part of the transformation. From Eq. (5),

$$\mathbf{r}^5_{\mathscr{E}} = \mathbf{P}(\mathbf{r}_1 + t_1\dot{\mathbf{r}}_1),$$

$$\dot{\mathbf{r}}^5_{\mathscr{E}} = \dot{\mathbf{P}}(\mathbf{r}_1 + t_1\dot{\mathbf{r}}_1) + \mathbf{P}\dot{\mathbf{r}}_1.$$

Eliminating $\mathbf{r}_1$ from this pair of equations, and using Eq. (7) to express $\dot{\mathbf{r}}^4_{\mathscr{E}}$ in a similar manner, gives

$$\dot{\mathbf{r}}^5_{\mathscr{E}} = \dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} + \mathbf{P}\dot{\mathbf{r}}_1,$$

$$\dot{\mathbf{r}}^4_{\mathscr{E}} = \dot{\mathbf{P}}_{\rm o}\mathbf{P}_{\rm o}^{-1}\mathbf{r}^4_{\mathscr{E}} + \mathbf{P}_{\rm o}\dot{\mathbf{r}}_0.$$

Using the above equations and the relationship between $\dot{\mathbf{r}}^5_{\mathscr{E}}$ and $\dot{\mathbf{r}}^4_{\mathscr{E}}$ given in Eq. (6), we have

$$\mathbf{P}\dot{\mathbf{r}}_1 + \dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{r}^5_{\mathscr{E}} = F\left[-\dot{E}\frac{\partial}{\partial E}\mathbf{R}_3(-E)\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\dot{\mathbf{P}}_{\rm o}\mathbf{P}_{\rm o}^{-1}\mathbf{r}^4_{\mathscr{E}} + \mathbf{R}_3(-E)\mathbf{P}_{\rm o}\dot{\mathbf{r}}_0\right]. \tag{8}$$

The matrix $\dot{\mathbf{P}}\mathbf{P}^{-1}$ is given by

$$\dot{\mathbf{P}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & -m & -n \\ m & 0 & -l \\ n & l & 0 \end{bmatrix},$$

where $l = -\dot{\theta}_{\rm A}\sin(-z_{\rm A}) - \dot{\zeta}_{\rm A}\cos(-z_{\rm A})\sin\theta_{\rm A}$, $m = \dot{z}_{\rm A} + \dot{\zeta}_{\rm A}\cos\theta_{\rm A}$, and $n = \dot{\theta}_{\rm A}\cos(-z_{\rm A}) - \dot{\zeta}_{\rm A} \times \sin(-z_{\rm A})\sin\theta_{\rm A}$. At epoch $\mathscr{E}$ = 1984 January 1.0 in the FK5 system $l = 1'' \times 10^{-9}$ per Julian century. On the right-hand side of the equation, in the FK4 system at epoch $\mathscr{E}$ = 1984 January 1.0, the numerical value of $l_{\rm o}$ in the matrix $\dot{\mathbf{P}}_{\rm o}\mathbf{P}_{\rm o}^{-1}$ is $l_{\rm o} = 5'' \times 10^{-8}$ per tropical century.

The vector equation (8) represents three scalar equations at epoch $\mathscr{E}$. By multiplying these equations by $x^5_{\mathscr{E}}, y^5_{\mathscr{E}}, z^5_{\mathscr{E}}$ as appropriate and combining them to form μ and μ' (see Sec. IV), we obtain

$$\mu + m + n\sin\alpha\tan\delta - l\cos\alpha\tan\delta = (\mu_{\rm o} + m_{\rm o} + n_{\rm o}\sin\alpha_{\rm o}\tan\delta_{\rm o} - l_{\rm o}\cos\alpha_{\rm o}\tan\delta_{\rm o})F$$

$$\mu' + n\cos\alpha + l\sin\alpha = (\mu'_{\rm o} + n_{\rm o}\cos\alpha_{\rm o} + l_{\rm o}\sin\alpha_{\rm o})F,$$

where $\alpha = \alpha_{\rm o} + E$ and $\delta = \delta_{\rm o}$ are the right ascension and declination at epoch $\mathscr{E}$ on the FK5 and FK4 systems. These equations are identical to the equations used in the classical method (Paper I) except for the terms in l and $l_{\rm o}$, which are too small to be included in the classical formulas.

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