

## PSEUDO-HIGH-ORDER SYMPLECTIC INTEGRATORS

J. E. CHAMBERS

Armagh Observatory, Armagh, Northern Ireland, BT61 9DG, UK; jec@star.arm.ac.uk

AND

M. A. MURISON

Astronomical Applications Department, US Naval Observatory, 3450 Massachusetts Avenue, NW, Washington, DC 20392-5420;  
murison@aa.usno.navy.mil

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### ABSTRACT

Symplectic  $N$ -body integrators are widely used to study problems in celestial mechanics. The most popular algorithms are of second and fourth order, requiring two and six substeps per time step, respectively. The number of substeps increases rapidly with order in time step, rendering higher order methods impractical. However, symplectic integrators are often applied to systems in which perturbations between bodies are a small factor  $\epsilon$  of the force due to a dominant central mass. In this case, it is possible to create optimized symplectic algorithms that require fewer substeps per time step. This is achieved by only considering error terms of order  $\epsilon$  and neglecting those of order  $\epsilon^2$ ,  $\epsilon^3$ , etc. Here we devise symplectic algorithms with four and six substeps per step which effectively behave as fourth- and sixth-order integrators when  $\epsilon$  is small. These algorithms are more efficient than the usual second- and fourth-order methods when applied to planetary systems.

*Key words:* celestial mechanics, stellar dynamics — methods:  $n$ -body simulations — methods: numerical

### 1. INTRODUCTION

Symplectic integrators are widely used to study problems in celestial mechanics. These integrators have two advantages over most other algorithms. First, they exhibit no long-term build up in energy error. Second, the motion of each object about the central body can be “built in,” so that the choice of step size,  $\tau$ , is determined by the perturbations between bodies, whose magnitude is a factor  $\epsilon$  smaller than the forces due to the central body (Wisdom & Holman 1991).

The most popular algorithm is the second-order symplectic integrator. The error at each step is proportional to  $\epsilon\tau^3$ , so that the likely error for an integration as a whole is  $\sim\epsilon\tau^2$ . The second-order method is easy to implement, consisting of only two substeps, including one force evaluation, per time step. It is also very fast for integrations requiring moderate accuracy.

For more accurate integrations, it is better to use the fourth-order method (Forest & Ruth 1990). Here the error at each step is proportional to  $\epsilon\tau^5$ , although each step is computationally more expensive since it consists of six substeps. Yoshida (1990) has developed sixth- and eighth-order symplectic integrators. However, these do not appear to be competitive in most situations, due to the large number of substeps required.

Here we show how to construct what are effectively high-order (fourth, sixth, etc.) symplectic integrators that require fewer substeps per time step than those in current use. The trick is to take into account the dependence of each error term on  $\epsilon$  when choosing the coefficients for each substep. The algorithms are designed by eliminating error terms proportional to  $\epsilon$  up to the desired order of the time step. Error terms proportional to  $\epsilon^2$ ,  $\epsilon^3$ , etc., in low orders of the time step, still exist. However, in many situations these terms are negligible, and the integrators behave as if they are of higher order than the leading error term in  $\tau$  suggests.

Section 2 gives a quick review of how symplectic integrators are traditionally constructed using Lie algebra. In § 3, we show how to build more efficient symplectic algorithms using fewer substeps. Section 4 contains results of test integrations that compare the new algorithms with traditional symplectic integrators.

### 2. SYMPLECTIC INTEGRATORS

Symplectic integrators for the  $N$ -body problem can be constructed starting from Hamilton’s equations of motion:

$$\begin{aligned}\frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i},\end{aligned}\tag{1}$$

where  $x_i$  and  $p_i$  are the coordinates and momenta of each body, respectively, and  $H$  is the Hamiltonian of the system.

Using these equations, the rate of change of any dynamical quantity  $q(x, p, t)$  can be expressed as

$$\frac{dq}{dt} = \sum_{i=1}^{3N} \left( \frac{\partial q}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial q}{\partial p_i} \frac{\partial H}{\partial x_i} \right) \equiv \{q, H\} \equiv Fq,\tag{2}$$

where  $\{, \}$  are Poisson brackets and  $F$  is a differential operator.

The formal solution of equation (2) is

$$q(t) = e^{\tau F} q(t - \tau) = \left( 1 + \tau F + \frac{\tau^2 F^2}{2} + \dots \right) q(t - \tau),$$

where  $F^2 q = F(Fq)$ , etc.

Now suppose that we are able to split the Hamiltonian into two pieces,  $H_A$  and  $H_B$ , so that each part of the problem can be solved relatively easily in the absence of the other. The solution for  $q$  becomes

$$q(t) = e^{\tau(A+B)} q(t - \tau), \tag{3}$$

where  $A$  and  $B$  are differential operators related to  $H_A$  and  $H_B$ , respectively, in the same way that  $F$  is related to  $H$ .

The Baker-Campbell-Hausdorff (BCH) formula states that, for any noncommutative operators  $A$  and  $B$ ,

$$\exp(A) \cdot \exp(B) = \exp(C),$$

where  $C$  is a series consisting of nested commutators,

$$C = A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, A, B] + \frac{1}{12} [B, B, A] + \dots,$$

where the commutator  $[A, B] = AB - BA \neq 0$  in general (see, for example, Yoshida 1990 or Forest & Ruth 1990). Here we have used the nested commutator notation  $[A, B, C] = [A, [B, C]]$ , etc.

Hence, if we evolve  $q$  under the two parts of the Hamiltonian separately, one after the other, we have

$$\exp(\tau A) \cdot \exp(\tau B) q(t - \tau) = \exp \left[ \tau F + \frac{\tau^2}{2} [A, B] + \dots \right] q(t - \tau). \tag{4}$$

This is identical to the right-hand side of equation (3) to  $O(\tau)$ , and so equation (4) represents a first-order integrator. Each step of the integrator consists of two substeps, with the whole step giving an error of  $O(\tau^2)$ . Alternatively, we can say that the integrator exactly solves a problem whose Hamiltonian is given by

$$H_{\text{integ}} = H + \frac{\tau}{2} \{H_B, H_A\} + O(\tau^2)$$

(see, for example, Saha & Tremaine 1992). Provided that  $\tau$  is small, and  $\{H_B, H_A\}$  remains bounded, the energy of the integrated system will always be near to that of the real system.

Other integrators can be found by combining exponential operators in such a way that they are equivalent to equation (3) up to a given order in  $\tau$ . For example, we have the second-order symplectic integrator

$$\begin{aligned} \text{S2A} &= \exp \left( \frac{\tau}{2} A \right) \cdot \exp(\tau B) \cdot \exp \left( \frac{\tau}{2} A \right) \\ &= \exp \left[ \tau F + \frac{\tau^3}{12} [B, B, A] - \frac{\tau^3}{24} [A, A, B] + \dots \right]. \end{aligned}$$

When many integration steps are performed one after another, the  $\exp(\tau A/2)$  terms at the end of one step and the start of another can be combined. Hence, the second-order integrator also consists of only two substeps, except at the beginning and the end of an integration.

Another second-order integrator is

$$\begin{aligned} \text{S2B} &= \exp \left( \frac{\tau}{2} B \right) \cdot \exp(\tau A) \cdot \exp \left( \frac{\tau}{2} B \right) \\ &= \exp \left[ \tau F + \frac{\tau^3}{12} [A, A, B] - \frac{\tau^3}{24} [B, B, A] + \dots \right]. \end{aligned} \tag{5}$$

The distinction between S2A and S2B (which at first sight appear to be the same) will become apparent in the next section, when we consider situations in which  $A$  and  $B$  are of different magnitude.

Forest & Ruth (1990) give a fourth-order symplectic integrator with six substeps per step:

$$\begin{aligned} \text{S4B} &= \exp \left( \frac{\tau B}{2c} \right) \cdot \exp \left( \frac{\tau A}{c} \right) \cdot \exp \left[ \frac{\tau B(1-k)}{2c} \right] \cdot \exp \left( \frac{-\tau k A}{c} \right) \cdot \exp \left[ \frac{\tau B(1-k)}{2c} \right] \\ &\quad \cdot \exp \left( \frac{\tau A}{c} \right) \cdot \exp \left( \frac{\tau B}{2c} \right) \\ &= \exp [\tau F + O(\tau^5)], \end{aligned}$$

where  $k = 2^{1/3}$  and  $c = 2 - k$ . Note that the middle three substeps move in the opposite direction to the integration as a whole.

Higher order integrators require progressively more substeps. Yoshida (1990) gives examples of sixth- and eighth-order integrators using 14 and 30 substeps, respectively. In the next section, we will show how to create what are *effectively* fourth- and sixth-order integrators (and in principle, eighth-order, etc.) using fewer substeps than are required conventionally.

### 3. CONSTRUCTING PSEUDO-ORDER INTEGRATORS

Up to this point we have not considered the details of how  $H$  is split. Suppose that one part of the Hamiltonian is much smaller than the other, i.e.,  $H = H_A + \epsilon H_B$ , where  $\epsilon \ll 1$ . Now consider the error terms in the second-order integrator of equation (5):

$$S2B = \exp \left[ \tau F + \frac{\epsilon \tau^3}{12} [A, A, B] - \frac{\epsilon^2 \tau^3}{24} [B, B, A] + \dots \right].$$

One of the  $O(\tau^3)$  error terms is smaller than the other by a factor of  $\epsilon$ .

Similarly, for the fourth-order integrator:

$$S4B = \exp [\tau F + O(\epsilon \tau^5) + O(\epsilon^2 \tau^5) + O(\epsilon^3 \tau^5) + O(\epsilon^4 \tau^5)].$$

Some of these error terms are insignificant compared to others, and yet this was not taken into account when constructing the integrator. The only design criterion was that S4B should contain no error terms below the fifth power in the time step. If we take into account the dependence of the error terms on both  $\tau$  and  $\epsilon$ , we can design more efficient symplectic integrators.

To construct the new integrators, we again employ the BCH formula. Adapting the expression for the BCH formula given by Yoshida (1990), we have

$$\begin{aligned} & \exp(a_1 \tau A) \cdot \exp(b_1 \epsilon \tau B) \\ &= \exp \left[ (a_1 A + \epsilon b_1 B) \tau + \epsilon \tau^2 \left( \frac{a_1 b_1}{2} \right) [A, B] + \epsilon \tau^3 \left( \frac{a_1^2 b_1}{12} \right) [A, A, B] + \epsilon^2 \tau^3 \left( \frac{a_1 b_1^2}{12} \right) [B, B, A] \right] \\ & \quad + \left[ \epsilon^2 \tau^4 \left( \frac{a_1^2 b_1^2}{24} \right) [A, B, B, A] - \epsilon \tau^5 \left( \frac{a_1^4 b_1}{720} \right) [A, A, A, A, B] - \epsilon^4 \tau^5 \left( \frac{a_1 b_1^4}{720} \right) [B, B, B, B, A] + \dots \right], \end{aligned}$$

where  $a_1$  and  $b_1$  are constants. Additional fifth-order commutators are present; however, we will only require terms that contain either  $A$  or  $B$  once, since these are the type of error term we are seeking to eliminate.

Applying the BCH formula twice, Yoshida (1990) gives an expression for a symmetric product of three exponential operators:

$$\begin{aligned} & \exp(b_1 \epsilon \tau B) \cdot \exp(a_1 \tau A) \cdot \exp(b_1 \epsilon \tau B) \\ &= \exp \left[ (a_1 A + 2\epsilon b_1 B) \tau + \epsilon \tau^3 \left( \frac{a_1^2 b_1}{6} \right) [A, A, B] - \epsilon^2 \tau^3 \left( \frac{a_1 b_1^2}{6} \right) [B, B, A] \right] \\ & \quad - \left[ \epsilon \tau^5 \left( \frac{a_1^4 b_1}{360} \right) [A, A, A, A, B] + \epsilon^4 \tau^5 \left( \frac{7a_1 b_1^4}{360} \right) [B, B, B, B, A] + \dots \right]. \end{aligned} \tag{6}$$

Again we have neglected fifth-order terms that contain both  $A$  and  $B$  more than once. Note that there are no terms containing even powers of the time step: Yoshida shows that this is a general property of any symmetric arrangement of exponential operators. From now on we will consider only symmetrical integrators because of this property.

We need to extend equation (6) once more to get a pseudo-fourth-order integrator, and twice more for a pseudo-sixth-order one. By substituting  $a_2 A$  for  $b_1 B$  in equation (6), and substituting the right-hand side of equation (6) for  $a_1 A$ , we get

$$\begin{aligned} & \exp(a_2 \tau A) \cdot \exp(b_1 \epsilon \tau B) \cdot \exp(a_1 \tau A) \cdot \exp(b_1 \epsilon \tau B) \cdot \exp(a_2 \tau A) \\ &= \exp \left\{ (a_1 + 2a_2) \tau A + 2b_1 \epsilon \tau B + \epsilon \tau^3 \left( \frac{b_1}{6} \right) [(a_1 + 2a_2)^2 - 6a_2(a_1 + a_2)] [A, A, B] \right\} \\ & \quad + \epsilon^2 \tau^3 \left( \frac{b_1^2}{6} \right) (4a_2 - a_1) [B, B, A] - \epsilon \tau^5 \left( \frac{b_1}{360} \right) [(a_1 + 2a_2)^4 - 30a_2^2(a_1 + a_2)^2] [A, A, A, A, B] \\ & \quad - \left\{ \epsilon^4 \tau^5 \left( \frac{b_1^4}{360} \right) (16a_2 - 7a_1) [B, B, B, B, A] + \dots \right\}. \end{aligned} \tag{7}$$

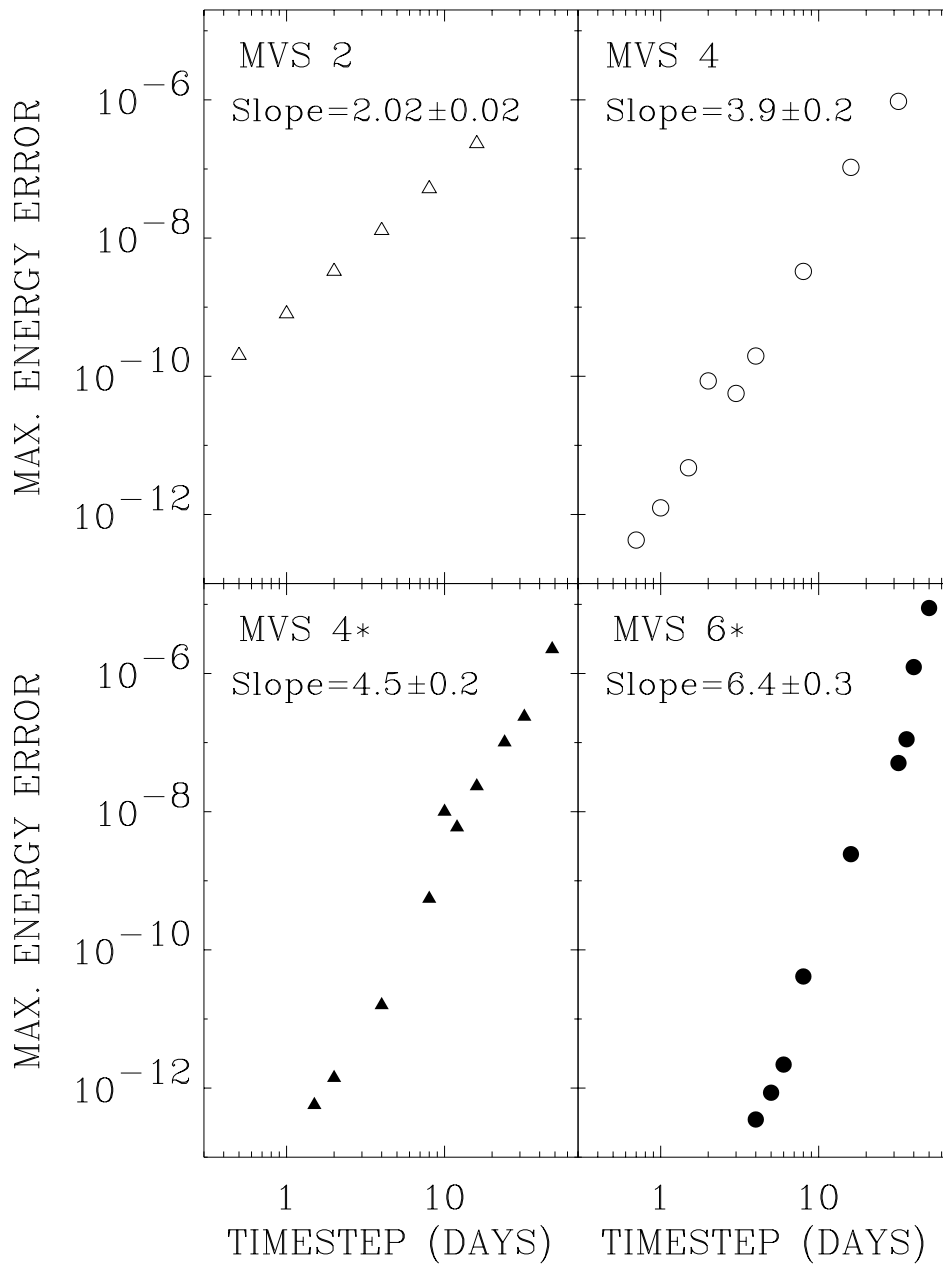


FIG. 1.—Maximum relative energy error vs. step size for a 10,000 yr integration of the four terrestrial planets using various symplectic integrators

Finally, substituting the right-hand side of equation (7) for  $a_1 A$  in equation (6), and replacing  $b_1 B$  with  $b_2 B$ , we arrive at

$$\begin{aligned}
 & \exp(b_2 \epsilon \tau B) \cdot \exp(a_2 \tau A) \cdot \exp(b_1 \epsilon \tau B) \cdot \exp(a_1 \tau A) \cdot \exp(b_1 \epsilon \tau B) \cdot \exp(a_2 \tau A) \cdot \exp(b_2 \epsilon \tau B) \\
 &= \exp \left\{ (a_1 + 2a_2)\tau A + 2(b_1 + b_2)\epsilon \tau B + \epsilon \tau^3 \left[ \frac{(b_1 + b_2)(a_1 + 2a_2)^2 - 6a_2 b_1 (a_1 + a_2)}{6} \right] [A, A, B] \right\} \\
 & \quad - \epsilon^2 \tau^3 \left[ \frac{(a_1 + 2a_2)(b_1 + b_2)^2 - 6a_2 b_1^2}{6} \right] [B, B, A] \\
 & \quad - \epsilon \tau^5 \left[ \frac{(b_1 + b_2)(a_1 + 2a_2)^4 - 30a_2^2 b_1 (a_1 + a_2)^2}{360} \right] [A, A, A, A, B] \\
 & \quad + \left\{ \epsilon^4 \tau^5 \left[ \frac{7(a_1 + 2a_2)(b_1 + b_2)^4 - 60a_2 b_1^2 (b_1 + b_2)^2 + 30a_2 b_1^4}{360} \right] [B, B, B, B, A] + \dots \right\}. \quad (8)
 \end{aligned}$$

The first stage in converting these general expressions into specific integrators is to make the coefficients of the linear  $A$  and  $B$  terms equal to 1. This places two constraints on the values of the constants. We can then get what is effectively a

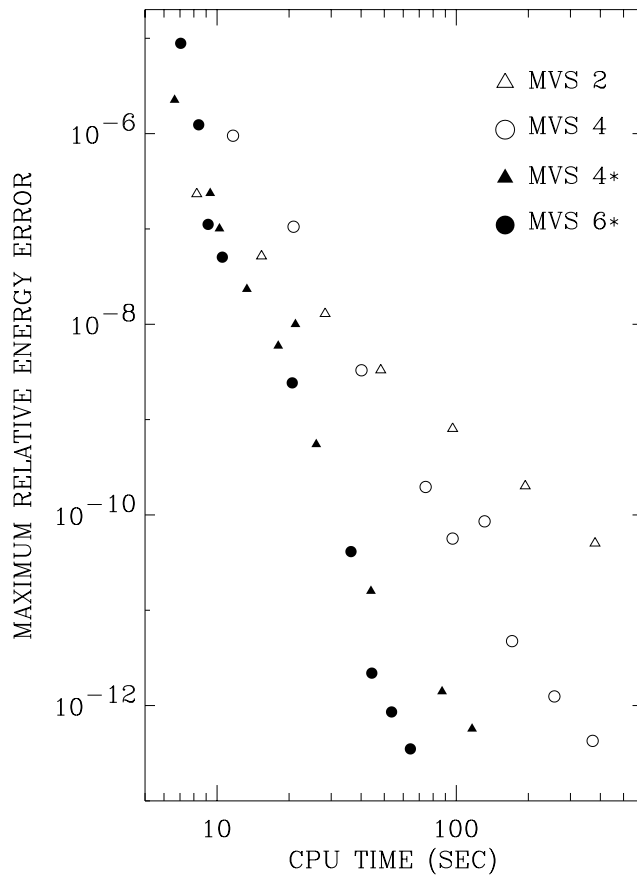


FIG. 2.—Maximum relative energy error vs. CPU time for a 10,000 yr integration of the four terrestrial planets using various symplectic integrators

fourth-order integrator by simply eliminating the  $[A, A, B]$  term from equation (7). The leading error terms will now be  $O(\epsilon^2\tau^3)$  and  $O(\epsilon\tau^5)$ . Provided that  $\epsilon$  is small enough, the largest error term will be  $O(\epsilon\tau^5)$ , and the integrator effectively will be of fourth order in the time step. Applying these conditions, we require

$$\begin{aligned} a_1 + 2a_2 &= 1, \\ 2b_1 &= 1, \\ 1 - 6a_2(1 - a_2) &= 0, \end{aligned} \tag{9}$$

where we have used the first two lines of equation (9) in deriving the third.

Alternatively, we may construct an integrator in which each step begins by advancing  $H_B$  instead of  $H_A$ . Unlike conventional symplectic integrators, such as  $S2A$  and  $S2B$ , we cannot use the same set of coefficients when exchanging  $A$  and  $B$ . Instead, we must derive a new set of coefficients by interchanging  $A$  and  $\epsilon B$  in equation (7) and then eliminating the new  $[A, A, B]$  term. When we do this, the first two lines of equation (9) remain as before, but the third expression becomes

$$6a_2 - 1 = 0. \tag{10}$$

To get a pseudo-sixth-order integrator, we eliminate terms containing  $[A, A, B]$  and  $[A, A, A, A, B]$ . This will produce an extra constraining equation, so we need an extra constant. We get this by using an integrator with the form of equation (8) instead of equation (7). The corresponding equations for the constants are

$$\begin{aligned} a_1 + 2a_2 &= 1, \\ 2(b_1 + b_2) &= 1, \\ 1/2 - 6a_2 b_1(1 - a_2) &= 0, \\ 1/2 - 30a_2^2 b_1(1 - a_2)^2 &= 0. \end{aligned} \tag{11}$$

If we prefer an integration step that begins by advancing  $H_A$ , we can interchange  $A$  and  $\epsilon B$  in equation (8) and eliminate the new  $[A, A, B]$  and  $[A, A, A, A, B]$  terms. In this case, the last two lines of equation (11) become

$$\begin{aligned} 1/4 - 6a_2 b_1^2 &= 0, \\ 7/16 - 15a_2 b_1^2 + 30a_2 b_1^4 &= 0. \end{aligned} \tag{12}$$

The leading error terms for each of these integrators are  $O(\epsilon^2\tau^3)$  and  $O(\epsilon\tau^7)$ . The latter will be dominant if  $\epsilon$  is small enough, so that the algorithms behave as sixth-order integrators.

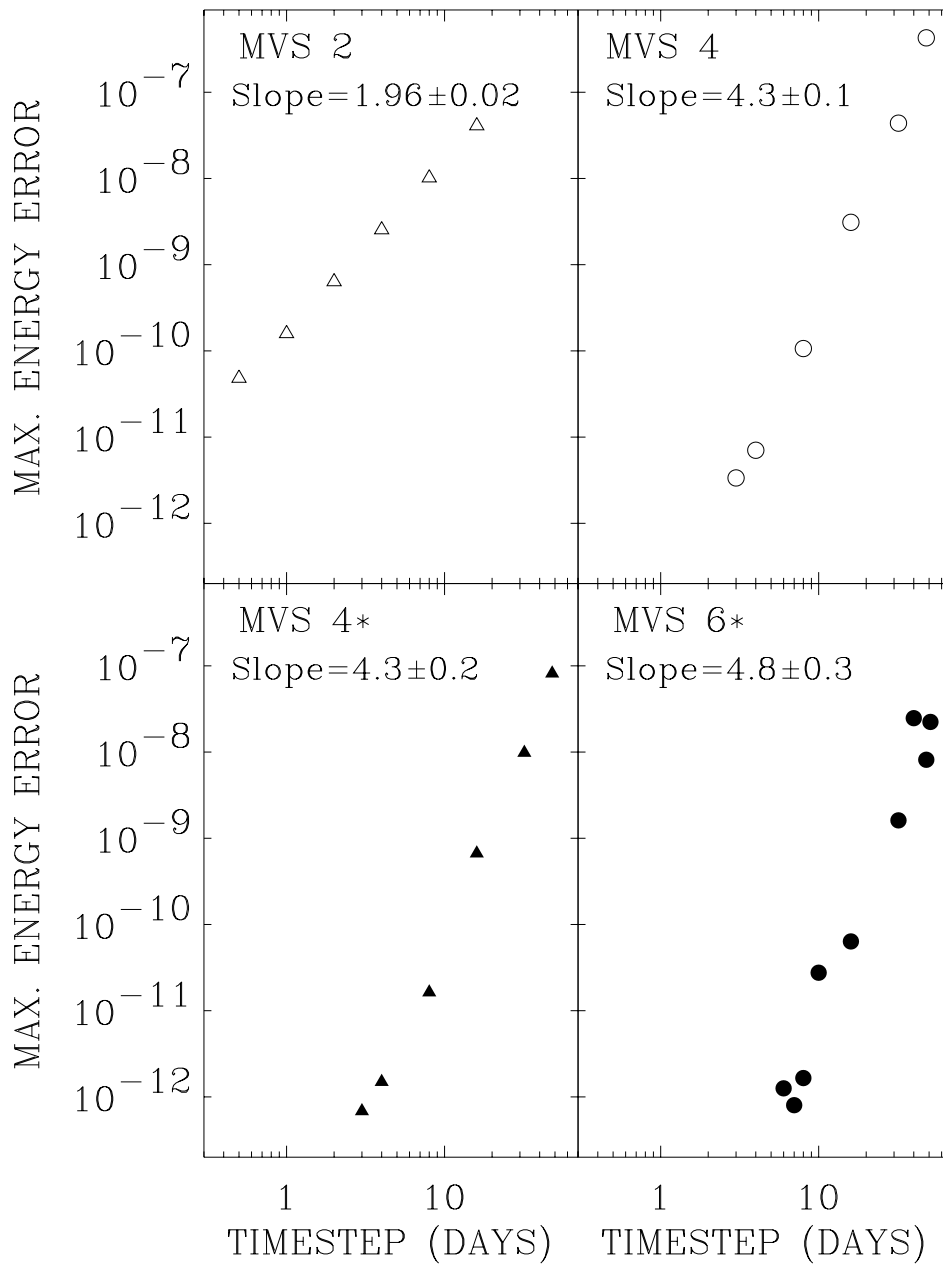


FIG. 3.—Maximum relative energy error vs. step size for a 10,000 yr integration of the nine planets using various symplectic integrators

### 3.1. Pseudo-Fourth- and Sixth-Order Examples

Solving equations (9) and (10), we obtain two pseudo-fourth-order integrators:

$$\begin{aligned}
 \text{S4A*} &= \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right] \cdot \exp\left(\frac{\epsilon\tau B}{2}\right) \cdot \exp\left(\frac{\tau A}{\sqrt{3}}\right) \cdot \exp\left(\frac{\epsilon\tau B}{2}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{3}}\right)\right] \\
 &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{2 - \sqrt{3}}{24}\right)[B, B, A] - \frac{\epsilon\tau^5}{4320}[A, A, A, A, B] + \dots\right], \\
 \text{S4B*} &= \exp\left(\frac{\epsilon\tau B}{6}\right) \cdot \exp\left(\frac{\tau A}{2}\right) \cdot \exp\left(\frac{2\epsilon\tau B}{3}\right) \cdot \exp\left(\frac{\tau A}{2}\right) \cdot \exp\left(\frac{\epsilon\tau B}{6}\right) \\
 &= \exp\left[\tau F + \frac{\epsilon^2\tau^3}{72}[B, B, A] + \frac{\epsilon\tau^5}{2880}[A, A, A, A, B] + \dots\right],
 \end{aligned}$$

where the asterisk in S4A\* indicates that it only behaves as a fourth-order integrator for small values of  $\epsilon$ .

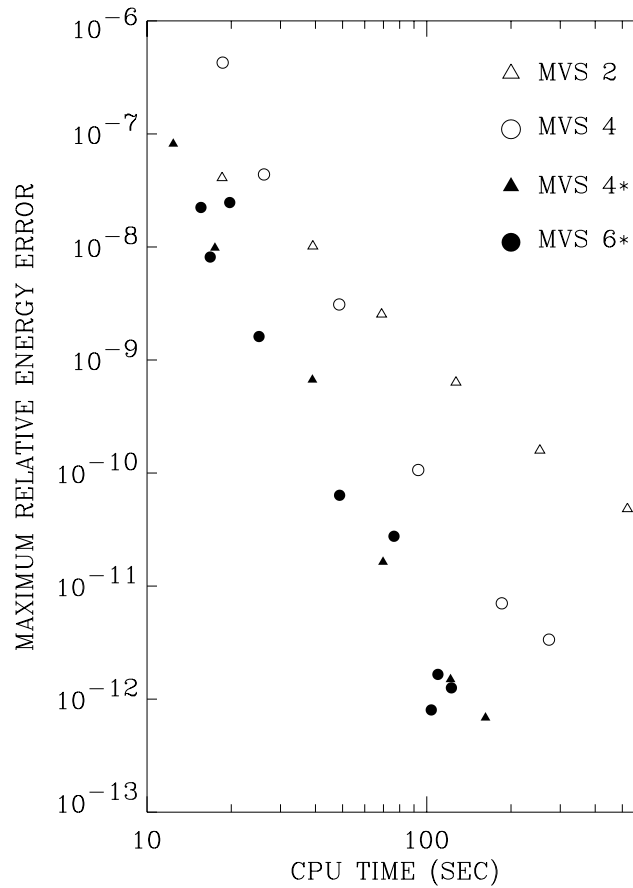


FIG. 4.—Maximum relative energy error vs. CPU time for a 10,000 yr integration of the nine planets using various symplectic integrators

Equations (11) and (12) give two pseudo-sixth-order integrators:

$$\begin{aligned}
 S6A^* &= \exp\left[\frac{\tau A}{2}\left(1 - \frac{3}{\sqrt{15}}\right)\right] \cdot \exp\left(\frac{5\epsilon\tau B}{18}\right) \cdot \exp\left(\frac{3\tau A}{2\sqrt{15}}\right) \cdot \exp\left(\frac{4\epsilon\tau B}{9}\right) \cdot \exp\left(\frac{3\tau A}{2\sqrt{15}}\right) \\
 &\quad \cdot \exp\left(\frac{5\epsilon\tau B}{18}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{3}{\sqrt{15}}\right)\right] \\
 &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{54 - 13\sqrt{15}}{648}\right)[B, B, A] + O(\epsilon\tau^7)\right], \\
 S6B^* &= \exp\left(\frac{\epsilon\tau B}{12}\right) \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right] \cdot \exp\left(\frac{5\epsilon\tau B}{12}\right) \cdot \exp\left(\frac{\tau A}{\sqrt{5}}\right) \cdot \exp\left(\frac{5\epsilon\tau B}{12}\right) \\
 &\quad \cdot \exp\left[\frac{\tau A}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right] \cdot \exp\left(\frac{\epsilon\tau B}{12}\right) \\
 &= \exp\left[\tau F + \epsilon^2\tau^3\left(\frac{13 - 5\sqrt{5}}{288}\right)[B, B, A] + O(\epsilon\tau^7)\right].
 \end{aligned}$$

McLachlan (1995) has independently derived similar solutions.

Unlike the fourth-order algorithm of Forest & Ruth (1990) and the sixth-order integrators of Yoshida (1990), the algorithms above contain no substeps that move in the opposite direction to the main integration. An additional solution exists for each of equations (9), (11), and (12); however, these have error terms with larger numerical coefficients than the integrators we show here.

The same method can be used to generate a pseudo-eighth-order integrator and so on. Each higher order will require only two more substeps than the previous one, since only one more commutator needs to be eliminated in each case. For example, to create a pseudo-eighth-order integrator requires the elimination of the  $[A, A, A, A, A, A, B]$  term in addition to those that are absent from the pseudo-sixth-order case. However, depending on the system to be integrated, there will come a point at which the  $\epsilon^2\tau^3$  error term becomes the most important. In principle, one could devise another set of integrators that

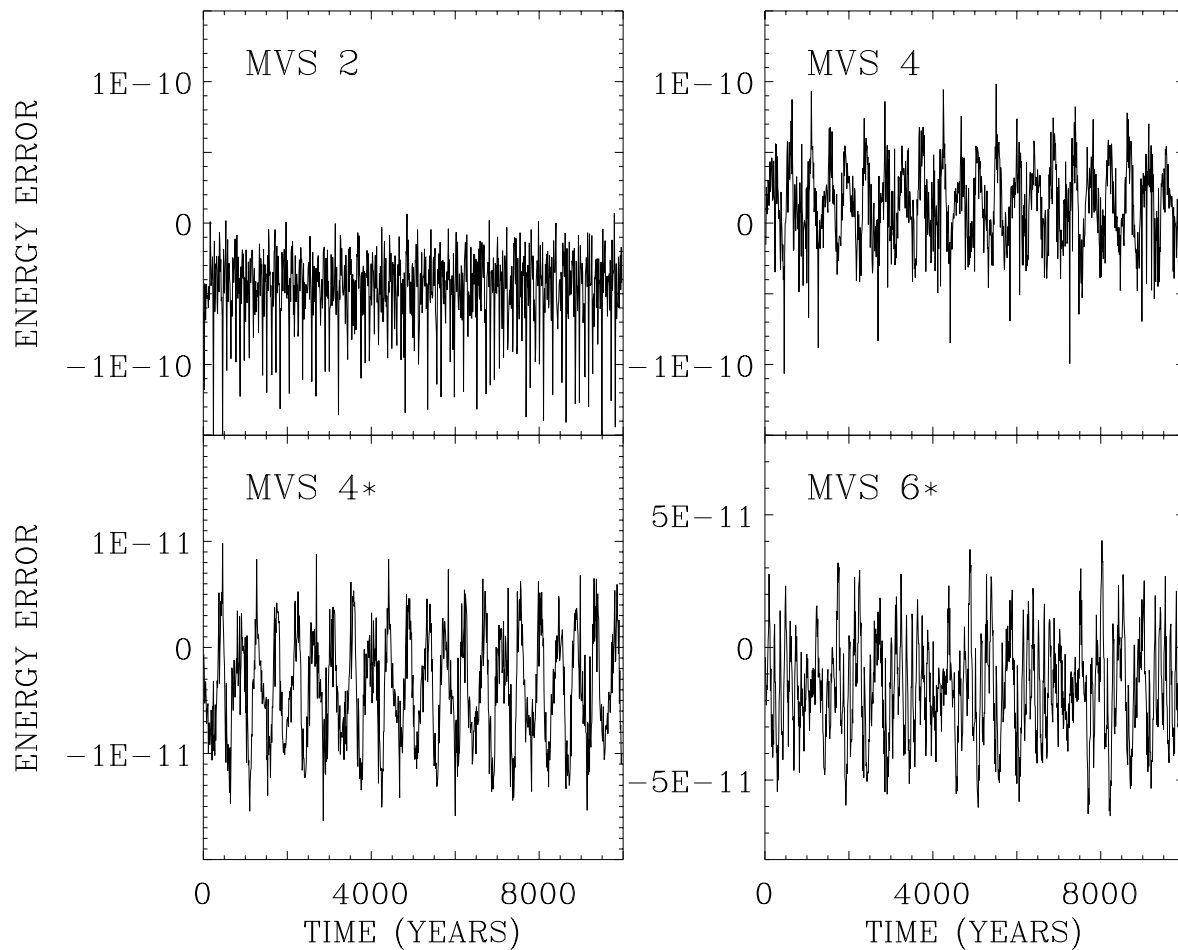


FIG. 5.—Relative energy error vs. time for 10,000 yr integrations of the nine planets using various symplectic integrators

eliminates terms in  $\epsilon^2\tau^m$  for small  $m$ , in addition to terms in  $\epsilon\tau^m$ . However, achieving each new order will generally require the elimination of more than one commutator term, so that these integrators increase in complexity much more rapidly than those described here.

Murison & Chambers (1999) have independently derived the two fourth-order integrators above, among others, using a symbolic algebra package. Further results from that approach will follow in another paper. We note that the pseudo-order algorithms can be adapted to use independent time steps for each planet (cf. Saha & Tremaine 1994), or to include close encounters (Duncan, Levison, & Lee 1998; Chambers 1999).

#### 4. NUMERICAL COMPARISONS

In this section, we test the pseudo-fourth- and sixth-order integrators derived in § 3 against the well-known second- and fourth-order symplectic algorithms. We use the “mixed-variable” method of Wisdom & Holman (1991), in which the Hamiltonian is divided into a Keplerian part,  $H_K$ , and an interaction part,  $H_I$ . Under  $H_K$ , each object moves on an unperturbed Keplerian orbit about the central body. Under  $H_I$ , each object remains fixed while receiving an impulse due to the gravitational perturbations of all the other objects except the central body. As suggested by Wisdom and Holman, we use Jacobi coordinates rather than barycentric coordinates. The integrations themselves were carried out using a modified version of the *Mercury* integrator package (Chambers & Migliorini 1997).

The pseudo-order integrators require that the ratio  $\epsilon = H_I/H_K \ll 1$ . In our first test, we integrate the orbits of the four inner planets of the solar system in the absence of the outer planets. In this case  $\epsilon \sim 10^{-5}$ . Figure 1 shows the results of a 10,000 yr integration using the conventional second- and fourth-order symplectic integrators, S2B and S4B, and the pseudo-order integrators S4B\* and S6B\*. For each integration, the maximum relative energy error is shown as a function of the step size.

For the second- and fourth-order integrators, the maximum energy error is roughly proportional to  $\tau^2$  and  $\tau^4$ , respectively, where  $\tau$  is the time step. This is what we would expect to find. For the pseudo-fourth- and sixth-order integrators, the maximum energy error varies as  $\tau^4$  and  $\tau^6$ . That is, they behave as fourth- and sixth-order integrators, as we anticipated, despite the fact that they contain error terms of lower order in the time step.

Using the mean relative energy error per integration instead of the maximum error gives results similar to Figure 1. The corresponding slopes are  $2.10 \pm 0.05$  for S2B,  $3.9 \pm 0.3$  for S4B,  $4.6 \pm 0.3$  for S4B\*, and  $6.4 \pm 0.4$  for S6B\*.



Figure 2 shows the amount of CPU time required for the integrations shown in Figure 1. For energy errors of 1 part in  $10^6$  or  $10^8$  there is not much to choose between the four algorithms. For higher levels of accuracy, S4B outperforms S2B. However, the pseudo-integrators S4B\* and S6B\* do even better. At an accuracy of 1 part in  $10^{10}$ , they are roughly an order of magnitude faster than the conventional second-order integrator, and 3 times faster than the fourth-order integrator. For accuracies of better than  $10^{-11}$ , S6B\* shows greater performance than S4B\*.

The pseudo-fourth-order integrator is more efficient than the real fourth-order integrator for two reasons. It requires fewer substeps per time step, and it has a slightly smaller leading error term.

As a more interesting test, we integrated the whole planetary system (Mercury to Pluto) for 10,000 yr. Figure 3 shows the energy-error results of these integrations. The behavior of S2B, S4B and S4B\* is similar to that for the integrations of the terrestrial planets. However, the energy error for S6B\* varies roughly as  $\tau^5$  rather than  $\tau^6$ . It is not obvious why this should be, although the difference from the terrestrial-planet integration (Fig. 1) is presumably due to the fact that  $\epsilon$  is 2 orders of magnitude larger in this case.

Figure 4 shows the CPU time required for the integrations of the nine planets. The results are similar to the integration of the inner planets, except that S6B\* has only a marginal advantage over S4B\* at the highest levels of accuracy.

Since writing the original draft of this manuscript, we have become aware of the symplectic corrector method of Wisdom, Holman, & Touma (1996), which substantially improves the efficiency of the second-order symplectic integrator. We present the pseudo-order integrators as an alternative strategy for designing accurate algorithms. It is possible to devise other symplectic correctors using the same approach we use in § 3 to design the integrator kernel: that is, by considering the dependence of the resulting error terms on  $\epsilon$  as well as  $\tau$  (Mikkola 1997; Rauch & Holman 1999). Finally, we suggest that it may be possible to design symplectic correctors to improve the performance of pseudo-order algorithms, since the pseudo-order methods exhibit similar high-frequency oscillations in energy error to the second- and fourth-order symplectic integrators (see Fig. 5).

In summary, we conclude that the new pseudo-order integrators outperform the widely used second- and fourth-order algorithms at all reasonable values of the energy error, for problems involving a dominant central mass.

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